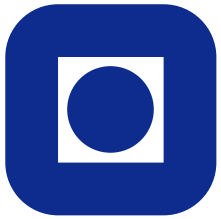

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TMA4115 - Calculus 3

Lecture 24, April 11

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Spring 2013

Yesterday's lecture

Yesterday we studied

- real matrices with complex eigenvalues,
- systems of first order differential equations.

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Today's lecture

Today we shall continue studying systems of first order differential equations.

Systems of first order differential equations

A *system of first order differential equations with constant coefficients* is a system of differential equations

$$\begin{aligned} x_1' &= a_{11}x_1 + \cdots + a_{1n}x_n \\ x_2' &= a_{21}x_1 + \cdots + a_{2n}x_n \\ &\vdots \\ x_n' &= a_{n1}x_1 + \cdots + a_{nn}x_n \end{aligned}$$

where $x_1, \ldots, x_n$ are differentiable functions of t with derivatives $x_1', \ldots, x_n'$, and the a_{ij} are constants.

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Systems of first order differential equations

If we let

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} \quad \mathbf{x}'(t) = \begin{bmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{bmatrix} \text{ and } A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}$$

then we write the system as $\mathbf{x}'(t) = A\mathbf{x}(t)$.

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The general solution to a system of first order differential equations

If an $n \times n$ matrix A has n linear independent eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$, then the general solution of the system $\mathbf{x}'(t) = A\mathbf{x}(t)$ is

$$\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + \cdots + c_n\mathbf{v}_n e^{\lambda_n t}$$

where $c_1, \ldots, c_n$ are constants.

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Attractors, repellers and saddle points

Let $\mathbf{x}'(t) = A\mathbf{x}(t)$ be a system of first order differential equations.

- The origin is called an *attractor* (or a *sink*) if every solution of $\mathbf{x}'(t) = A\mathbf{x}(t)$ converges to $\mathbf{0}$ as $t \to \infty$.
- The origin is called a *repeller* (or a *source*) if every solution of $\mathbf{x}'(t) = A\mathbf{x}(t)$ converges to $\mathbf{0}$ as $t \to -\infty$.
- The origin is called a *saddle point* if there is a solution $\mathbf{x}_1$ of $\mathbf{x}'(t) = A\mathbf{x}(t)$ that converges to $\mathbf{0}$ as $t \to \infty$ and a solution $\mathbf{x}_2$ of $\mathbf{x}'(t) = A\mathbf{x}(t)$ that converges to $\mathbf{0}$ as $t \to -\infty$.

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Example

Let us find the general solution to the system $\mathbf{x}'(t) = A\mathbf{x}(t)$ where $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$, and determine if the origin is an attractor, repeller, or a saddle point.

Solution

The characteristic polynomial of A is

$$\det(A - \lambda I_2) = \begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 9 = \lambda^2 - 2\lambda - 8,$$

so the eigenvalues of A are $\frac{2 \pm \sqrt{2^2 - 4(-8)}}{2} = \frac{2 \pm 6}{2} = \begin{cases} 4 \\ -2 \end{cases}$.

$A - 4I_2 = \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix}$, so we see that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to 4.

$A + 2I_2 = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$, so we see that $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector corresponding to -2.

It follows that the general solution to the system $\mathbf{x}'(t) = A\mathbf{x}(t)$ is $\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t}$.

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Solution (cont.)

Since the solution $\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} \to \mathbf{0}$ as $t \to -\infty$, and the solution $\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t} \to \mathbf{0}$ as $t \to \infty$, it follows that the origin is a saddle point.

Decoupling a dynamical system

- Consider a system $\mathbf{x}'(t) = A\mathbf{x}(t)$ of first order differential equations where A is an $n \times n$ matrix with n linearly independent eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$.
- Let $P = [\mathbf{v}_1 \ \ldots \mathbf{v}_n]$ and let D be the diagonal matrix with entries $\lambda_1, \ldots, \lambda_n$. Then $A = PDP^{-1}$.
- If we let $\mathbf{y}(t) = P^{-1}\mathbf{x}(t)$, then $\mathbf{x}'(t) = A\mathbf{x}(t)$ if and only if

$$\begin{aligned}\mathbf{y}'(t) &= \frac{d}{dt}(P^{-1}\mathbf{x}(t)) = P^{-1}\mathbf{x}'(t) = P^{-1}A\mathbf{x}(t) \\ &= P^{-1}(PDP^{-1})\mathbf{x}(t) = DP^{-1}\mathbf{x}(t) = D\mathbf{y}(t).\end{aligned}$$

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Decoupling a dynamical system (cont.)

- The general solution to the system $\mathbf{y}'(t) = D\mathbf{y}(t)$ is $\mathbf{y}(t) = c_1\mathbf{e}_1 e^{\lambda_1 t} + \cdots + c_n\mathbf{e}_n e^{\lambda_n t}$
- It follows that the general solution to $\mathbf{x}'(t) = A\mathbf{x}(t)$ is $\mathbf{x}(t) = P\mathbf{y}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + \cdots + c_n\mathbf{v}_n e^{\lambda_n t}$.

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Example

Let us find the general solution to the system $\mathbf{x}'(t) = A\mathbf{x}(t)$ where $A = \begin{bmatrix} 4 & -2 \\ 1 & 6 \end{bmatrix}$.

Solution

The eigenvalues of A are $5 \pm i$.

$\begin{bmatrix} 1-i \\ -1 \end{bmatrix}$ is an eigenvector corresponding to $5+i$.

$\begin{bmatrix} 1+i \\ -1 \end{bmatrix}$ is an eigenvector corresponding to $5-i$.

Let $\mathbf{x}_1(t) = \begin{bmatrix} 1-i \\ -1 \end{bmatrix} e^{(5+i)t}$ and $\mathbf{x}_2(t) = \begin{bmatrix} 1+i \\ -1 \end{bmatrix} e^{(5-i)t}$.

Then $\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)$ is the general solution of $\mathbf{x}'(t) = A\mathbf{x}(t)$.

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Solution (cont.)

Let
$\mathbf{y}_1(t) = \text{Re}(\mathbf{x}_1(t)) = \text{Re}\left(\begin{bmatrix} 1-i \\ -1 \end{bmatrix} e^{(5+i)t}\right) = e^{5t}\begin{bmatrix} \cos t + \sin t \\ -\cos t \end{bmatrix}$
and $\mathbf{y}_2(t) = \text{Im}(\mathbf{x}_1(t)) = \text{Re}\left(\begin{bmatrix} 1-i \\ -1 \end{bmatrix} e^{(5+i)t}\right) =$
$e^{5t}\begin{bmatrix} -\cos t + \sin t \\ -\sin t \end{bmatrix}$.
Then $\mathbf{y}_1'(t) = \text{Re}(\mathbf{x}_1'(t)) = \text{Re}(A\mathbf{x}_1(t)) = A\mathbf{y}_1(t)$, and $\mathbf{y}_2'(t) = \text{Im}(\mathbf{x}_1'(t)) = \text{Im}(A\mathbf{x}_1(t)) = A\mathbf{y}_2(t)$, so $\mathbf{y}_1(t)$ and $\mathbf{y}_2(t)$ are solutions of $\mathbf{x}'(t) = A\mathbf{x}(t)$.
Since $\mathbf{y}_1$ and $\mathbf{y}_2$ are linearly independent, it follows that $\{\mathbf{y}_1, \mathbf{y}_2\}$ is a basis for the set of solutions of $\mathbf{x}'(t) = A\mathbf{x}(t)$.

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Solution (cont.)

So the general solution to $\mathbf{x}'(t) = A\mathbf{x}(t)$ is

$$\mathbf{x}(t) = c_1\mathbf{y}_1(t) + c_2\mathbf{y}_2(t) =$$

$$c_1 e^{5t}\begin{bmatrix}\cos t + \sin t \\ -\cos t\end{bmatrix} + c_2 e^{5t}\begin{bmatrix}-\cos t + \sin t \\ -\sin t\end{bmatrix}.$$

Complex eigenvalues

Let A be a real 2×2 matrix with a complex eigenvalue λ and corresponding eigenvector $\mathbf{v}$. Let

$$\mathbf{y}_1(t) = \big((\mathrm{Re}(\mathbf{v}))\cos(\mathrm{Im}(\lambda)t) - (\mathrm{Im}(\mathbf{v}))\sin(\mathrm{Im}(\lambda)t)\big)e^{\mathrm{Re}(\lambda)t}$$

$$\mathbf{y}_2(t) = \big((\mathrm{Re}(\mathbf{v}))\sin(\mathrm{Im}(\lambda)t) + (\mathrm{Im}(\mathbf{v}))\cos(\mathrm{Im}(\lambda)t)\big)e^{\mathrm{Re}(\lambda)t}$$

Then $\mathbf{x}(t) = c_1\mathbf{y}_1(t) + c_2\mathbf{y}(t)$ is the general solution of $\mathbf{x}'(t) = A\mathbf{x}(t)$.

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Writing second-order differential equations as systems of first order differential equations

- Let $y'' + ay' + by = 0$ be a homogeneous second-order differential equation with constant coefficients.
- Let $x_1(t) = y(t)$ and $x_2(t) = y'(t)$.
- Then the equation $y'' + ay' + by = 0$ is equivalent to the system

$$x_1' = x_2$$
$$x_2' = -bx_1 - ax_2$$

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Example

Let us find the solution to the initial values problem

$$y'' - y' - 6y = 0,\ y(0) = 3,\ y'(0) = 4.$$

Solution

Let $x_1(t) = y(t)$ and $x_2(t) = y'(t)$. Then the equation $y'' - y' - 6y = 0$ is equivalent to the system

$$\begin{aligned} x_1'(t) &= x_2(t) \\ x_2'(t) &= 6x_1(t) + x_2(t) \end{aligned}$$

Let $A = \begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix}$. Then the eigenvalues of A are -2 and 3, $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is an eigenvector corresponding to -2, and $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is an eigenvalue corresponding to 3.

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Solution (cont.)

It follows that $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{3t}$ is the general solution to the system

$$x_1'(t) = x_2(t)$$
$$x_2'(t) = 6x_1(t) + x_2(t)$$

Let $\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{3t}$. Then

$\mathbf{x}(0) = c_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ -2c_1 + 3c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ if and only if $c_1 = 1$ and $c_2 = 2$. It follows that $y(t) = e^{-2t} + 2e^{3t}$ is the solution to the initial value problem
$y'' - y' - 6y = 0,\ y(0) = 3,\ y'(0) = 4.$

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Problem 6 from August 2011

Two tanks contain salt water. Tank 1 holds 200 litres and tank 2 holds 100 litres. Pure water enters tank 1 at a rate of 4 litres per second, salt water flows from tank 1 to tank 2 at the rate of 9 litres per seconds, salt water flows from tank 2 to tank 1 at the rate of 5 litres per second, and salt water flows from tank 2 and out of the system at a rate of 4 litres per second. The tanks are stirred so that the salt in each is evenly distributed.

1. Find a system of differential equations which describes the amount of salt $y_1(t)$ and $y_2(t)$ in, respectively, tank 1 and tank 2.
2. Determine $y_1(t)$ and $y_2(t)$ given that at the start ($t = 0$), tank 1 contained 100 grammes of salt and tank 2 contained 200 grammes of salt.

Solution

Notice that the volume of water in each of the two tanks is constant. It follows that $y_1(t)$ and $y_2(t)$ satisfy the equations

$$y_1'(t) = \frac{-9}{200} y_1(t) + \frac{5}{100} y_2(t)$$
$$y_2'(t) = \frac{9}{200} y_1(t) + \frac{-9}{100} y_2(t)$$

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Solution (cont.)

Let $A = \frac{1}{200}\begin{bmatrix} -9 & 10 \\ 9 & -18 \end{bmatrix}$. Then

$$\begin{aligned} \det(A - \lambda I_2) &= \left(\frac{1}{200}\right)^2 \begin{vmatrix} -9 - 200\lambda & 10 \\ 9 & -18 - 200\lambda \end{vmatrix} \\ &= \frac{1}{40000}\big((-9 - 200\lambda)(-18 - 200\lambda) - 90\big) \\ &= \frac{1}{40000}(\eta^2 + 27\eta + 72) \end{aligned}$$

where $\eta = 200\lambda$. The solution to $\eta^2 + 27\eta + 72 = 0$ is

$\eta = \frac{-27 \pm \sqrt{27^2 - 4 \cdot 72}}{2} = \frac{-27 \pm 21}{2} = \begin{cases} -3 \\ -24 \end{cases}$, so the eigenvalues of A are $\frac{-3}{200}$ and $\frac{-24}{200}$.

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Solution (cont.)

$A + \frac{3}{200} I_2 = \frac{1}{200} \begin{bmatrix} -6 & 10 \\ 9 & -15 \end{bmatrix}$ so $\begin{bmatrix} 10 \\ 6 \end{bmatrix}$ is an eigenvalue of A corresponding to the eigenvalue $\frac{-3}{200}$.

$A + \frac{24}{200} I_2 = \frac{1}{200} \begin{bmatrix} 15 & 10 \\ 9 & 6 \end{bmatrix}$ so $\begin{bmatrix} 10 \\ -15 \end{bmatrix}$ is an eigenvalue of A corresponding to the eigenvalue $\frac{-24}{200}$.

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Solution (cont.)

It follows that $\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = c_1 \begin{bmatrix} 10 \\ 6 \end{bmatrix} e^{\frac{-3}{200}t} + c_2 \begin{bmatrix} 10 \\ -15 \end{bmatrix} e^{\frac{-24}{200}t}$ is the general solution to the system

$$y_1'(t) = \frac{-9}{200} y_1(t) + \frac{5}{100} y_2(t)$$
$$y_2'(t) = \frac{9}{200} y_1(t) + \frac{-9}{100} y_2(t)$$

We have that

$$10c_1 + 10c_2 = y_1(0) = 100$$

and that

$$6c_1 - 15c_2 = y_2(0) = 200.$$

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Solution (cont.)

We reduce the augmented matrix of the system

$$\begin{aligned} 10c_1 + 10c_2 &= 100 \\ 6c_1 - 15c_2 &= 200 \end{aligned}$$

to its reduce echelon form.

$$\begin{bmatrix} 10 & 10 & 100 \\ 6 & -15 & 200 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 10 \\ 6 & -15 & 200 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 10 \\ 0 & -21 & 140 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 1 & 10 \\ 0 & 1 & -20/3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 50/3 \\ 0 & 1 & -20/3 \end{bmatrix}$$

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Solution (cont.)

We see that $c_1 = 50/3$, $c_2 = -20/3$ is the unique solution to the system

$$\begin{aligned} 10c_1 + 10c_2 &= 100 \\ 6c_1 - 15c_2 &= 200 \end{aligned}$$

Thus, $y_1(t) = 10c_1 e^{\frac{-3}{200}t} + 2c_2 e^{\frac{-24}{200}t} = \frac{500}{3} e^{\frac{-3}{200}t} - \frac{200}{3} e^{\frac{-24}{200}t}$ and $y_2(t) = 6c_1 e^{\frac{-3}{200}t} - 3c_2 e^{\frac{-24}{200}t} = 100e^{\frac{-3}{200}t} + 100e^{\frac{-24}{200}t}$.

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Problem 8 from June 2012

Let $A = \begin{bmatrix} -2 & -5 \\ 5 & -2 \end{bmatrix}$.

1. Find the complex eigenvalues of A and the corresponding eigenvectors in $\mathbb{C}^2$.
2. Find the solutions of the system of differential equations $\mathbf{y}'(t) = A\mathbf{y}(t)$ satisfying $\mathbf{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

   The answer should be written in the form $\mathbf{y}(t) = e^{\lambda t} \begin{bmatrix} a\cos(\omega t) + b\sin(\omega t) \\ c\cos(\omega t) + d\sin(\omega t) \end{bmatrix}$.

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Solution

$$\det(A-\lambda I_2) = \begin{vmatrix} -2-\lambda & -5 \\ 5 & -2-\lambda \end{vmatrix} = (-2-\lambda)^2 + 25$$
$$= \lambda^2 + 4\lambda + 29$$

so the eigenvalues of A are
$\frac{-4\pm\sqrt{4^2-4\cdot 29}}{2} = \frac{-4\pm\sqrt{-100}}{2} = \frac{-4\pm 10i}{2} = -2\pm 5i$.
$A-(-2+5i) = \begin{bmatrix} -5i & -5 \\ 5 & -5i \end{bmatrix}$, so $\begin{bmatrix} 1 \\ -i \end{bmatrix}$ is an eigenvector of A corresponding to the eigenvalue $= -2+5i$.

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Solution (cont.)

It follows that $\begin{bmatrix} 1 \\ i \end{bmatrix}$ is an eigenvector of A corresponding to the eigenvalue $= -2 - 5i$.
Since the eigenvalues of A each has multiplicity 1, it follows that the corresponding eigenspaces each has dimension 1, so the eigenvectors of A corresponding to the the eigenvalue $= -2 + 5i$ are the nonzero multiple of $\begin{bmatrix} 1 \\ -i \end{bmatrix}$, and the eigenvectors of A corresponding to the the eigenvalue $= -2 - 5i$ are the nonzero multiple of $\begin{bmatrix} 1 \\ i \end{bmatrix}$.

Solution (cont.)

The general solution to $\mathbf{y}'(t) = A\mathbf{y}(t)$ is

$$\mathbf{y}(t) = c_1 \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{(-2+5i)t} + c_2 \begin{bmatrix} 1 \\ i \end{bmatrix} e^{(-2-5i)t}.$$

If $\mathbf{y}(t) = c_1 \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{(-2+5i)t} + c_2 \begin{bmatrix} 1 \\ i \end{bmatrix} e^{(-2-5i)t}$, then

$$\mathbf{y}(0) = c_1 \begin{bmatrix} 1 \\ -i \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ i(-c_1 + c_2) \end{bmatrix}.$$

It follows that

$$\mathbf{y}(t) = 1/2 \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{(-2+5i)t} + 1/2 \begin{bmatrix} 1 \\ i \end{bmatrix} e^{(-2-5i)t} = e^{-2t} \begin{bmatrix} \cos(5t) \\ \sin(5t) \end{bmatrix}$$

is the solutions of the system of differential equations

$\mathbf{y}'(t) = A\mathbf{y}(t)$ satisfying $\mathbf{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

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Plan for next week

Wednesday we shall introduce and study

- *the inner product*,
- the *length* of a vector,
- *orthogonality* and *orthogonal* sets in $\mathbb{R}^n$.

Sections 6.1–6.2 in “Linear Algebras and Its Applications” (pages 329-346).

Thursday we shall introduce and study

- *orthogonal projections*,
- the *Gram-Schmidt process*,
- *QR factorization*.

Sections 6.3–6.4 in “Linear Algebras and Its Applications” (pages 347-360).

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