

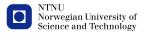
TMA4115 - Calculus 3 Lecture 23, April 10

Toke Meier Carlsen Norwegian University of Science and Technology Spring 2013

Review of last week's lecture

Last week we introduce and studied

- *eigenvectors*, *eigenvalues* and *eigenspaces* of square matrices,
- the characteristic polynomial of a square matrix,
- diagonal matrices,
- how to *diagonalizable* a matrix.



Today's lecture

Today we shall study

- real matrices with complex eigenvalues,
- systems of first order differential equations.



Eigenvalues and eigenvectors

Let *A* be an $n \times n$ matrix. Recall that:

- A scalar λ is an eigenvalue of A if and only if λ is a zero of the characteristic polynomial det(A – λI_n) of A.
- If λ is an eigenvalue of A, then an eigenvector of A corresponding to λ is a nontrivial solution to the equation Ax = λx.



Complex eigenvalues and eigenvectors

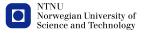
- A real matrix A might have a complex eigenvalue λ in which case the corresponding eigenvectors are alsocomplex.
- If λ is an eigenvalue of A with corresponding eigenvector
 v, then λ is an eigenvalue of A with corresponding eigenvector v.



Let
$$A = \begin{bmatrix} 4 & -2 \\ 1 & 6 \end{bmatrix}$$
. Then

$$\det(A - \lambda I_2) = \begin{vmatrix} 4 - \lambda & -2 \\ 1 & 6 - \lambda \end{vmatrix} = (4 - \lambda)(6 - \lambda) + 2 = \lambda^2 - 10\lambda + 26$$

and since the zeros of $\lambda^2 - 10\lambda + 26$ are $\lambda = \frac{10 \pm \sqrt{10^2 - 4 \cdot 26}}{2} = \frac{10 \pm \sqrt{-4}}{2} = \frac{10 \pm 2i}{2} = 5 \pm i$, it follows that the eigenvalues of *A* are $5 \pm i$.



Example (cont.)

 $A - (5+i)I_2 = \begin{bmatrix} 4 - (5+i) & -2 \\ 1 & 6 - (5+i) \end{bmatrix} = \begin{bmatrix} -1-i & -2 \\ 1 & 1-i \end{bmatrix},$ so $\mathbf{v}_1 = \begin{bmatrix} 1-i \\ -1 \end{bmatrix}$ is an eigenvalue of *A* corresponding to the eigenvalue $\lambda_1 = 5 + i$. It follows that $\mathbf{v}_2 = \begin{bmatrix} 1+i \\ -1 \end{bmatrix}$ is an eigenvalue of *A* corresponding to the eigenvalue $\lambda_2 = 5 - i$.



Let $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ where *a* and *b* are real numbers not both equal to 0. Then

$$\det(C-\lambda I_2) = \begin{vmatrix} a-\lambda & -b \\ b & a-\lambda \end{vmatrix} = (a-\lambda)^2 + b^2 = \lambda^2 - 2a\lambda + a^2 + b^2,$$

and since the zeros of $\lambda^2 - 2a\lambda + a^2 + b^2$ are $\lambda = \frac{2a \pm \sqrt{4a^2 - (a^2 + b^2)}}{2} = \frac{2a \pm \sqrt{-b^2}}{2} = \frac{2a \pm 2bi}{2} = a \pm bi$, it follows that the eigenvalues of *C* are $a \pm bi$. Let $r = |a + bi| = \sqrt{a^2 + b^2}$ and $\theta = \operatorname{Arg}(a + bi)$. Then $a = r \cos \theta$ and $b = r \sin \theta$, so $\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = C$. Norweging University of

Example (cont.)

It follows that the linear transformation $\mathbf{x} \mapsto C\mathbf{x}$ may be viewed as the composition of a rotation by $\operatorname{Arg}(a+bi)$ followed by a scaling by |a+bi|.



Rotation and scaling in \mathbb{R}^2

Let $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ where *a* and *b* are real numbers which are not both zero, and let $r = |a + ib| = \sqrt{a^2 + b^2}$ and $\theta = \operatorname{Arg}(a + ib)$. Then $C = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$ and the transformation $\mathbf{x} \mapsto C\mathbf{x}$ may be viewed as the composition of a rotation by $\operatorname{Arg}(a + bi)$ followed by a scaling by |a + bi|.



Let us return to the example where $A = \begin{bmatrix} 4 & -2 \\ 1 & 6 \end{bmatrix}$. We have seen that the eigenvalues of A are $5 \pm i$, and that $\mathbf{v}_1 = \begin{vmatrix} 1 - i \\ -1 \end{vmatrix}$ is an eigenvalue of A corresponding to the eigenvalue $\lambda_1 = 5 + i$, and that $\mathbf{v}_2 = \begin{vmatrix} 1 + i \\ -1 \end{vmatrix}$ is an eigenvalue. of A corresponding to the eigenvalue $\lambda_2 = 5 - i$. Let $P = [\operatorname{Re}(\mathbf{v}_1) \operatorname{Im}(\mathbf{v}_1)] = \begin{vmatrix} 1 & -1 \\ -1 & 0 \end{vmatrix}$ and $C = \begin{bmatrix} \operatorname{Re}(\lambda_1) & \operatorname{Im}(\lambda_1) \\ -\operatorname{Im}(\lambda_1) & \operatorname{Re}(\lambda_1) \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ -1 & 5 \end{bmatrix}.$



Example (cont.)

Then *P* is invertible, and

$$AP = \begin{bmatrix} 4 & -2 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 6 & -4 \\ -5 & -1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 6 & -4 \\ -5 & -1 \end{bmatrix}, \text{ so } A = PCP^{-1}$$



Real 2×2 matrices with complex eigenvalues

Theorem 9

Let *A* be a real 2 × 2 matrix with a complex eigenvalue $\lambda = a - bi$ ($b \neq 0$) and an associated eigenvector **v** in \mathbb{C}^2 . Then

$$A = PCP^{-1}$$

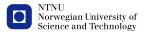
where $P = [\operatorname{Re}(\mathbf{v}) \operatorname{Im}(\mathbf{v})]$ and $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.



Problem 6 from the exam from December 2011

Let
$$A = \begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$$
.

- Show that $\mathbf{v} = \begin{bmatrix} 1 \\ i \end{bmatrix}$ is a (complex) eigenvector of *A*.
- Find the complex eigenvalues and complex eigenvectors of A.



Solution

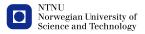
$$A\mathbf{v} = \begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} 1 - \sqrt{3}i \\ \sqrt{3} - i \end{bmatrix}$$
$$= (1 - \sqrt{3}i) \begin{bmatrix} 1 \\ i \end{bmatrix} = (1 - \sqrt{3}i)\mathbf{v}$$

(because $(1 - \sqrt{3}i)i = i - \sqrt{3}i^2 = \sqrt{3} - i$), so **v** is an eigenvector of *A* with corresponding eigenvalue $(1 - \sqrt{3}i)$. It follows that $\overline{\mathbf{v}} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$ is an eigenvector of *A* with corresponding eigenvalue $(1 + \sqrt{3}i)$.



Since A is a 2 × 2 matrix, it cannot have more than 2 eigenvalues, and the multiplicity of $(1 - \sqrt{3}i)$ and $(1 + \sqrt{3}i)$ must be 1.

It follows that the eigenvalues of *A* are $(1 - \sqrt{3}i)$ and $(1 + \sqrt{3}i)$ and that the eigenvectors of *A* are $\left\{t\begin{bmatrix}1\\i\end{bmatrix}: t \in \mathbb{C}, \ t \neq 0\right\}$ and $\left\{t\begin{bmatrix}1\\-i\end{bmatrix}: t \in \mathbb{C}, \ t \neq 0\right\}$.



Systems of first order differential equations

A system of first order differential equations with constant coefficients is a system of differential equations

$$\begin{aligned}
x'_{1} &= a_{11}x_{1} + \dots + a_{1n}x_{n} \\
x'_{2} &= a_{21}x_{1} + \dots + a_{2n}x_{n} \\
&\vdots \\
x'_{n} &= a_{n1}x_{1} + \dots + a_{nn}x_{n}
\end{aligned}$$

where x_1, \ldots, x_n are differentiable functions of *t* with derivatives x'_1, \ldots, x'_n , and the a_{ij} are constants.



NTNU Norwegian University of Science and Technology

$$\begin{aligned} x_1'(t) &= 5x_1(t) - x_2(t) \\ x_2'(t) &= -x_1(t) + 5x_2(t) \end{aligned}$$

is a system of first order differential equations with constant coefficients.



Systems of first order differential equations

If we let

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} \quad \mathbf{x}'(t) = \begin{bmatrix} x'_1(t) \\ \vdots \\ x'_n(t) \end{bmatrix} \text{ and } A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}$$

then we write the system as $\mathbf{x}'(t) = A\mathbf{x}(t)$.



The system

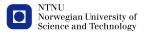
$$\begin{aligned} x_1'(t) &= 5x_1(t) - x_2(t) \\ x_2'(t) &= -x_1(t) + 5x_2(t) \end{aligned}$$

can be written as $\mathbf{x}'(t) = A\mathbf{x}(t)$ where $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ and $A = \begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix}$.



Systems of first order differential equations

- If u₁,..., u_k are solutions to the system x'(t) = Ax(t), then any linear combination c₁u₁ + ··· + c_ku_k of u₁,..., u_k is also a solution to x'(t) = Ax(t).
- The solution set of x'(t) = Ax(t) is an *n*-dimensional vector space, so if v₁,..., v_n are *n* linearly independent solutions to x'(t) = Ax(t), then any solution to x'(t) = Ax(t) is a linear combination of v₁,..., v_n.



Let us find the general solution to the system $\mathbf{x}'(t) = A\mathbf{x}(t)$ where $A = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$.



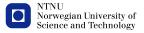
Solution

The equation $\mathbf{x}'(t) = A\mathbf{x}(t)$ is equivalent to the system

$$x'_1(t) = -x_1(t)$$

 $x'_2(t) = 4x_2(t)$

The solution to that system is $x_1(t) = c_1 e^{-t}$ and $x_2(t) = c_2 e^{4t}$, so the general solution to $\mathbf{x}'(t) = A\mathbf{x}(t)$ is $\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{4t}$.



Attractors, repellers and saddle points

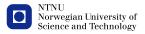
Let $\mathbf{x}'(t) = A\mathbf{x}(t)$ be a system of first order differential equations.

- The origin is called an *attractor* (or a *sink*) if every solution of x'(t) = Ax(t) converges to 0 as t → ∞.
- The origin is called a *repeller* (or a *source*) if every solution of x'(t) = Ax(t) converges to 0 as t → -∞.
- The origin is called a *saddle point* if there is a solution x₁ of x'(t) = Ax(t) that converges to 0 as t → ∞ and a solution x₂ of x'(t) = Ax(t) that converges to 0 as t → -∞.



Let us return to the system $\mathbf{x}'(t) = A\mathbf{x}(t)$ where $A = \begin{vmatrix} -1 & 0 \\ 0 & 4 \end{vmatrix}$.

We have seen that the general solution of the system is $\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{4t}.$ Since $\mathbf{x}_1(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-t} \to \mathbf{0}$ as $t \to \infty$, and $\mathbf{x}_2(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{4t} \to \mathbf{0}$ as $t \to -\infty$, the origin is a saddle point in this example.

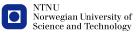


Let us find the general solution to the system $\mathbf{x}'(t) = A\mathbf{x}(t)$ that satisfies $\mathbf{x}(0) = \begin{bmatrix} 2\\ 4 \end{bmatrix}$, where $A = \begin{bmatrix} 5 & -1\\ -1 & 5 \end{bmatrix}$, and determine if the origin is an attractor, repeller, or a saddle point.



Solution

The eigenvalues of A are 4 and 6. $\begin{vmatrix} 1 \\ 1 \end{vmatrix}$ is an eigenvector corresponding to 4. $\begin{bmatrix} 1\\ -1 \end{bmatrix}$ is an eigenvector corresponding to 6. Let $\mathbf{x}_1(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}$ and $\mathbf{x}_2(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{6t}$. Then $A\mathbf{x}_{1}(t) = A \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} = 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} = \mathbf{x}'_{1}(t)$ and $A\mathbf{x}_{2}(t) = A\begin{bmatrix} 1\\-1 \end{bmatrix} e^{6t} = 6\begin{bmatrix} 1\\-1 \end{bmatrix} e^{6t} = \mathbf{x}_{2}'(t)$, so \mathbf{x}_{1} and \mathbf{x}_{2} are solutions to $\mathbf{x}'(t) = A\mathbf{x}(t)$



Suppose that $c_1 \mathbf{x}_1(t) + c_2 \mathbf{x}_2(t) = \mathbf{0}$. By evaluating at t = 0 we get $c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0$. Since $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ are linearly independent, it follows that $c_1 = c_2 = 0$. This shows that $\mathbf{x}_1(t)$ and $\mathbf{x}_{2}(t)$ are linearly independent. It follows that $\{\mathbf{x}_1(t), \mathbf{x}_2(t)\}\$ is a basis for the set of solutions of $\mathbf{x}'(t) = A\mathbf{x}(t)$, and thus that any solution of $\mathbf{x}'(t) = A\mathbf{x}(t)$ is a linear combination of $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$. Let $\mathbf{x}(t) = c_1 \mathbf{x}_1(t) + c_2 \mathbf{x}_2(t)$. Then $\mathbf{x}(0) = c_1 \mathbf{x}_1(0) + c_2 \mathbf{x}_2(0) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ c_1 - c_2 \end{bmatrix}$, so $\mathbf{x}(0) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ if and only if $c_1 + c_2 = 2$ and $c_1 - c_2 = 4$. Science and Technology

The solution to the system

$$c_1 + c_2 = 2$$
$$c_1 - c_2 = 4$$

is
$$c_1 = 3$$
, $c_2 = -1$, so
 $\mathbf{x}(t) = 3\mathbf{x}_1(t) - \mathbf{x}_2(t) = 3\begin{bmatrix}1\\1\end{bmatrix}e^{4t} - \begin{bmatrix}1\\-1\end{bmatrix}e^{6t}$ is a solution to
 $\mathbf{x}'(t) = A\mathbf{x}(t)$ that satisfies $\mathbf{x}(0) = \begin{bmatrix}2\\4\end{bmatrix}$.



Since the general solution of the system is $\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{6t}$ and $c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{6t} \rightarrow \mathbf{0}$ as $t \rightarrow -\infty$ for all c_1 and c_2 , we see that the origin is a repeller.

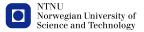


The general solution to a system of first order differential equations

If an $n \times n$ matrix A has n linear independent eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$, then the general solution of the system $\mathbf{x}'(t) = A\mathbf{x}(t)$ is

$$\mathbf{x}(t) = c_1 \mathbf{v}_1 e^{\lambda_1 t} + \cdots + c_n \mathbf{v}_n e^{\lambda_n t}$$

where c_1, \ldots, c_n are constants.



Tomorrow's lecture

Tomorrow we shall continue studying systems of first order differential equations. Section 5.7 in "Linear Algebras and Its Applications" (pages 295-301, and 311-319), and Section 4.2 in "Second-Order Equations" (pages xlv-xlix).

