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TMA4115 - Calculus 3

Lecture 23, April 10

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Spring 2013

Review of last week's lecture

Last week we introduce and studied

- *eigenvectors*, *eigenvalues* and *eigenspaces* of square matrices,
- the *characteristic polynomial* of a square matrix,
- *diagonal* matrices,
- how to *diagonalizable* a matrix.

Today's lecture

Today we shall study

- real matrices with complex eigenvalues,
- systems of first order differential equations.

Eigenvalues and eigenvectors

Let A be an $n \times n$ matrix. Recall that:

- A scalar λ is an eigenvalue of A if and only if λ is a zero of the characteristic polynomial $\det(A - \lambda I_n)$ of A.
- If λ is an eigenvalue of A, then an eigenvector of A corresponding to λ is a nontrivial solution to the equation $A\mathbf{x} = \lambda\mathbf{x}$.

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Complex eigenvalues and eigenvectors

- A real matrix A might have a complex eigenvalue λ in which case the corresponding eigenvectors are also complex.
- If λ is an eigenvalue of A with corresponding eigenvector $\mathbf{v}$, then $\overline{\lambda}$ is an eigenvalue of A with corresponding eigenvector $\overline{\mathbf{v}}$.

Example

Let $A = \begin{bmatrix} 4 & -2 \\ 1 & 6 \end{bmatrix}$. Then

$$\det(A - \lambda I_2) = \begin{vmatrix} 4-\lambda & -2 \\ 1 & 6-\lambda \end{vmatrix} = (4-\lambda)(6-\lambda)+2 = \lambda^2 - 10\lambda + 26$$

and since the zeros of $\lambda^2 - 10\lambda + 26$ are $\lambda = \frac{10 \pm \sqrt{10^2 - 4 \cdot 26}}{2} = \frac{10 \pm \sqrt{-4}}{2} = \frac{10 \pm 2i}{2} = 5 \pm i$, it follows that the eigenvalues of A are $5 \pm i$.

Example (cont.)

$$A-(5+i)I_2=\begin{bmatrix}4-(5+i) & -2\\ 1 & 6-(5+i)\end{bmatrix}=\begin{bmatrix}-1-i & -2\\ 1 & 1-i\end{bmatrix},$$

so $\mathbf{v}_1=\begin{bmatrix}1-i\\ -1\end{bmatrix}$ is an eigenvalue of A corresponding to the eigenvalue $\lambda_1=5+i$.

It follows that $\mathbf{v}_2=\begin{bmatrix}1+i\\ -1\end{bmatrix}$ is an eigenvalue of A corresponding to the eigenvalue $\lambda_2=5-i$.

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Example

Let $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ where a and b are real numbers not both equal to 0. Then

$$\det(C - \lambda I_2) = \begin{vmatrix} a-\lambda & -b \\ b & a-\lambda \end{vmatrix} = (a-\lambda)^2 + b^2 = \lambda^2 - 2a\lambda + a^2 + b^2,$$

and since the zeros of $\lambda^2 - 2a\lambda + a^2 + b^2$ are
$\lambda = \frac{2a \pm \sqrt{4a^2 - (a^2+b^2)}}{2} = \frac{2a \pm \sqrt{-b^2}}{2} = \frac{2a \pm 2bi}{2} = a \pm bi$, it follows that the eigenvalues of C are $a \pm bi$.
Let $r = |a + bi| = \sqrt{a^2 + b^2}$ and $\theta = \operatorname{Arg}(a + bi)$. Then $a = r\cos\theta$ and $b = r\sin\theta$, so

$$\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = C.$$

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Example (cont.)

It follows that the linear transformation $\mathbf{x} \mapsto C\mathbf{x}$ may be viewed as the composition of a rotation by $\operatorname{Arg}(a+bi)$ followed by a scaling by $|a+bi|$.

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Rotation and scaling in $\mathbb{R}^2$

Let $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ where a and b are real numbers which are not both zero, and let $r = |a + ib| = \sqrt{a^2 + b^2}$ and $\theta = \text{Arg}(a + ib)$.

Then $C = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and the transformation $\mathbf{x} \mapsto C\mathbf{x}$ may be viewed as the composition of a rotation by $\text{Arg}\,(a + bi)$ followed by a scaling by $|a + bi|$.

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Example

Let us return to the example where $A = \begin{bmatrix} 4 & -2 \\ 1 & 6 \end{bmatrix}$. We have seen that the eigenvalues of A are $5 \pm i$, and that $\mathbf{v}_1 = \begin{bmatrix} 1-i \\ -1 \end{bmatrix}$ is an eigenvalue of A corresponding to the eigenvalue $\lambda_1 = 5 + i$, and that $\mathbf{v}_2 = \begin{bmatrix} 1+i \\ -1 \end{bmatrix}$ is an eigenvalue of A corresponding to the eigenvalue $\lambda_2 = 5 - i$.

Let $P = [\operatorname{Re}(\mathbf{v}_1) \;\; \operatorname{Im}(\mathbf{v}_1)] = \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}$ and

$$C = \begin{bmatrix} \operatorname{Re}(\lambda_1) & \operatorname{Im}(\lambda_1) \\ -\operatorname{Im}(\lambda_1) & \operatorname{Re}(\lambda_1) \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ -1 & 5 \end{bmatrix}.$$

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Example (cont.)

Then P is invertible, and

$$AP = \begin{bmatrix} 4 & -2 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 6 & -4 \\ -5 & -1 \end{bmatrix} \text{ and}$$

$$\begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 6 & -4 \\ -5 & -1 \end{bmatrix}, \text{ so } A = PCP^{-1}.$$

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Real 2×2 matrices with complex eigenvalues

Theorem 9

Let A be a real 2×2 matrix with a complex eigenvalue $\lambda = a - bi$ ($b \neq 0$) and an associated eigenvector $\mathbf{v}$ in $\mathbb{C}^2$. Then

$$A = PCP^{-1}$$

where $P = [\mathrm{Re}(\mathbf{v}) \;\; \mathrm{Im}(\mathbf{v})]$ and $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

Problem 6 from the exam from December 2011

Let $A = \begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$.

1. Show that $\mathbf{v} = \begin{bmatrix} 1 \\ i \end{bmatrix}$ is a (complex) eigenvector of A.
2. Find the complex eigenvalues and complex eigenvectors of A.

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Solution

$$A\mathbf{v} = \begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} 1 - \sqrt{3}i \\ \sqrt{3} - i \end{bmatrix}$$
$$= (1 - \sqrt{3}i) \begin{bmatrix} 1 \\ i \end{bmatrix} = (1 - \sqrt{3}i)\mathbf{v}$$

(because $(1 - \sqrt{3}i)i = i - \sqrt{3}i^2 = \sqrt{3} - i$), so $\mathbf{v}$ is an eigenvector of A with corresponding eigenvalue $(1 - \sqrt{3}i)$.

It follows that $\bar{\mathbf{v}} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$ is an eigenvector of A with corresponding eigenvalue $(1 + \sqrt{3}i)$.

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Solution (cont.)

Since A is a 2×2 matrix, it cannot have more than 2 eigenvalues, and the multiplicity of $(1 - \sqrt{3}i)$ and $(1 + \sqrt{3}i)$ must be 1.

It follows that the eigenvalues of A are $(1 - \sqrt{3}i)$ and $(1 + \sqrt{3}i)$ and that the eigenvectors of A are

$$\left\{ t \begin{bmatrix} 1 \\ i \end{bmatrix} : t \in \mathbb{C},\ t \neq 0 \right\} \text{ and } \left\{ t \begin{bmatrix} 1 \\ -i \end{bmatrix} : t \in \mathbb{C},\ t \neq 0 \right\}.$$

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Systems of first order differential equations

A *system of first order differential equations with constant coefficients* is a system of differential equations

$$\begin{aligned} x_1' &= a_{11}x_1 + \cdots + a_{1n}x_n \\ x_2' &= a_{21}x_1 + \cdots + a_{2n}x_n \\ &\vdots \\ x_n' &= a_{n1}x_1 + \cdots + a_{nn}x_n \end{aligned}$$

where $x_1, \ldots, x_n$ are differentiable functions of t with derivatives $x_1', \ldots, x_n'$, and the a_{ij} are constants.

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Example

$$x_1'(t) = 5x_1(t) - x_2(t)$$
$$x_2'(t) = -x_1(t) + 5x_2(t)$$

is a system of first order differential equations with constant coefficients.

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Systems of first order differential equations

If we let

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} \quad \mathbf{x}'(t) = \begin{bmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{bmatrix} \text{ and } A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}$$

then we write the system as $\mathbf{x}'(t) = A\mathbf{x}(t)$.

Example

The system

$$x_1'(t) = 5x_1(t) - x_2(t)$$
$$x_2'(t) = -x_1(t) + 5x_2(t)$$

can be written as $\mathbf{x}'(t) = A\mathbf{x}(t)$ where $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ and $A = \begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix}$.

Systems of first order differential equations

- If $\mathbf{u}_1, \ldots, \mathbf{u}_k$ are solutions to the system $\mathbf{x}'(t) = A\mathbf{x}(t)$, then any linear combination $c_1\mathbf{u}_1 + \cdots + c_k\mathbf{u}_k$ of $\mathbf{u}_1, \ldots, \mathbf{u}_k$ is also a solution to $\mathbf{x}'(t) = A\mathbf{x}(t)$.
- The solution set of $\mathbf{x}'(t) = A\mathbf{x}(t)$ is an n-dimensional vector space, so if $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are n linearly independent solutions to $\mathbf{x}'(t) = A\mathbf{x}(t)$, then any solution to $\mathbf{x}'(t) = A\mathbf{x}(t)$ is a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

Example

Let us find the general solution to the system $\mathbf{x}'(t) = A\mathbf{x}(t)$ where $A = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$.

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Solution

The equation $\mathbf{x}'(t) = A\mathbf{x}(t)$ is equivalent to the system

$$\begin{aligned} x_1'(t) &= -x_1(t) \\ x_2'(t) &= 4x_2(t) \end{aligned}$$

The solution to that system is $x_1(t) = c_1 e^{-t}$ and $x_2(t) = c_2 e^{4t}$, so the general solution to $\mathbf{x}'(t) = A\mathbf{x}(t)$ is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{4t}.$$

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Attractors, repellers and saddle points

Let $\mathbf{x}'(t) = A\mathbf{x}(t)$ be a system of first order differential equations.

- The origin is called an *attractor* (or a *sink*) if every solution of $\mathbf{x}'(t) = A\mathbf{x}(t)$ converges to $\mathbf{0}$ as $t \to \infty$.
- The origin is called a *repeller* (or a *source*) if every solution of $\mathbf{x}'(t) = A\mathbf{x}(t)$ converges to $\mathbf{0}$ as $t \to -\infty$.
- The origin is called a *saddle point* if there is a solution $\mathbf{x}_1$ of $\mathbf{x}'(t) = A\mathbf{x}(t)$ that converges to $\mathbf{0}$ as $t \to \infty$ and a solution $\mathbf{x}_2$ of $\mathbf{x}'(t) = A\mathbf{x}(t)$ that converges to $\mathbf{0}$ as $t \to -\infty$.

Example

Let us return to the system $\mathbf{x}'(t) = A\mathbf{x}(t)$ where $A = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$.

We have seen that the general solution of the system is

$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{4t}$.

Since $\mathbf{x}_1(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-t} \to \mathbf{0}$ as $t \to \infty$, and $\mathbf{x}_2(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{4t} \to \mathbf{0}$ as $t \to -\infty$, the origin is a saddle point in this example.

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Example

Let us find the general solution to the system $\mathbf{x}'(t) = A\mathbf{x}(t)$ that satisfies $\mathbf{x}(0) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, where $A = \begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix}$, and determine if the origin is an attractor, repeller, or a saddle point.

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Solution

The eigenvalues of A are 4 and 6.

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to 4.

$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector corresponding to 6.

Let $\mathbf{x}_1(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}$ and $\mathbf{x}_2(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{6t}$. Then

$A\mathbf{x}_1(t) = A \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} = 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} = \mathbf{x}_1'(t)$ and

$A\mathbf{x}_2(t) = A \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{6t} = 6 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{6t} = \mathbf{x}_2'(t)$, so $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions to $\mathbf{x}'(t) = A\mathbf{x}(t)$.

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Solution (cont.)

Suppose that $c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = \mathbf{0}$. By evaluating at $t = 0$ we get $c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \mathbf{0}$. Since $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ are linearly independent, it follows that $c_1 = c_2 = 0$. This shows that $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are linearly independent. It follows that $\{\mathbf{x}_1(t), \mathbf{x}_2(t)\}$ is a basis for the set of solutions of $\mathbf{x}'(t) = A\mathbf{x}(t)$, and thus that any solution of $\mathbf{x}'(t) = A\mathbf{x}(t)$ is a linear combination of $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$.

Let $\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)$. Then

$\mathbf{x}(0) = c_1\mathbf{x}_1(0) + c_2\mathbf{x}_2(0) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ c_1 - c_2 \end{bmatrix}$, so

$\mathbf{x}(0) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ if and only if $c_1 + c_2 = 2$ and $c_1 - c_2 = 4$.

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Solution (cont.)

The solution to the system

$$\begin{aligned} c_1 + c_2 &= 2 \\ c_1 - c_2 &= 4 \end{aligned}$$

is $c_1 = 3$, $c_2 = -1$, so
$\mathbf{x}(t) = 3\mathbf{x}_1(t) - \mathbf{x}_2(t) = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} - \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{6t}$ is a solution to $\mathbf{x}'(t) = A\mathbf{x}(t)$ that satisfies $\mathbf{x}(0) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$.

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Solution (cont.)

Since the general solution of the system is $\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{6t}$ and $c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{6t} \to \mathbf{0}$ as $t \to -\infty$ for all c_1 and c_2, we see that the origin is a repeller.

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The general solution to a system of first order differential equations

If an $n \times n$ matrix A has n linear independent eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ with corresponding eigenvalues $\lambda_1, \dots, \lambda_n$, then the general solution of the system $\mathbf{x}'(t) = A\mathbf{x}(t)$ is

$$\mathbf{x}(t) = c_1 \mathbf{v}_1 e^{\lambda_1 t} + \cdots + c_n \mathbf{v}_n e^{\lambda_n t}$$

where $c_1, \dots, c_n$ are constants.

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Tomorrow’s lecture

Tomorrow we shall continue studying systems of first order differential equations. Section 5.7 in “Linear Algebras and Its Applications” (pages 295-301, and 311-319), and Section 4.2 in “Second-Order Equations” (pages xlv-xlix).

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