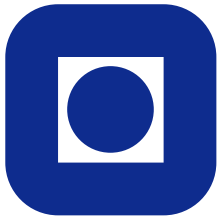

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TMA4115 - Calculus 3

Lecture 22, April 4

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Yesterday's lecture

Yesterday we introduced and studied

- *eigenvectors*, *eigenvalues* and *eigenspaces* of square matrices,
- the *characteristic polynomial* of a square matrix.

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Today's lecture

Today we shall look at

- *diagonal* matrices,
- how to *diagonalizable* a matrix.

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Eigenvectors, eigenvalues and eigenspaces

Let A be an $n \times n$ matrix. Recall that

- An *eigenvalue* of A is a scalar λ such that the equation $A\mathbf{x} = \lambda\mathbf{x}$ has a nontrivial solution.
- If λ is an eigenvalue of A, then a nontrivial solution to the equation $A\mathbf{x} = \lambda\mathbf{x}$ is called an *eigenvector (of A) corresponding to* λ.
- If λ is an eigenvalue of A, then the set of solutions to the equation $A\mathbf{x} = \lambda\mathbf{x}$ is called the *eigenspace (of A) corresponding to* λ.

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Finding a basis of an eigenspace

Let A be an $n \times n$ matrix and let λ be an eigenvalue of A. Recall that the eigenspace of A corresponding to λ is equal to $\text{Nul}(A - \lambda I_n)$, so we can find a basis for the eigenspace of A corresponding to λ by:

1. row reducing $A - \lambda I_n$ to its reduced echelon form,
2. write the solutions to the equation $(A - \lambda I_n)\mathbf{x} = \mathbf{0}$ (which is equivalent to the equation $A\mathbf{x} - \mathbf{0}$) as a linear combinations of vectors using the free variables as parameters.
3. Then these vectors form a basis of the eigenspace of A corresponding to λ.

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Finding the eigenvalues of a matrix

Let A be an $n \times n$ matrix. Recall that a scalar λ is an eigenvalue of A if and only if it is a zero of the *characteristic polynomial* $\det(A - \lambda I_n)$ of A.

Example

Let us find the eigenvalues of the matrix $A = \begin{bmatrix} 1 & -3 & 6 & 0 \\ 0 & -2 & 1 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 7 \end{bmatrix}$.

The characteristic polynomial of A is

$$\begin{aligned} \det(A - \lambda I_4) &= \begin{vmatrix} 1-\lambda & -3 & 6 & 0 \\ 0 & -2-\lambda & 1 & 1 \\ 0 & 0 & -\lambda & 2 \\ 0 & 0 & 0 & 7-\lambda \end{vmatrix} \\ &= (1-\lambda)(-2-\lambda)(-\lambda)(7-\lambda). \end{aligned}$$

It follows that the eigenvalues of A are 1, -2, 0 and 7.

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The eigenvalues of a triangular matrix

Theorem 1

The eigenvalues of a triangular matrix are the entries on its main diagonal.

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Example

Let $A = \begin{bmatrix} 1 & 2 & -1 \\ -5 & 6 & -3 \\ 5 & 2 & 11 \end{bmatrix}$, $D = \begin{bmatrix} 4 & -8 & 4 \\ 0 & 6 & 4 \\ 0 & 0 & 8 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$. Then P is invertible and $P^{-1}AP = D$.

Example (cont.)

Then $D - \lambda I_n = P^{-1}AP - \lambda P^{-1} I_n P = P^{-1}(A - \lambda I_n)P$. So

$$\begin{aligned}
\det(D - \lambda I_n) &= \det(P^{-1}(A - \lambda I_n)P) \\
&= \det(P^{-1}) \det(A - \lambda I_n) \det(P) \\
&= \det(P^{-1}) \det(P) \det(A - \lambda I_n) \\
&= \det(P^{-1} P) \det(A - \lambda I_n) \\
&= \det(I_n) \det(A - \lambda I_n) \\
&= \det(A - \lambda I_n).
\end{aligned}$$

It follows that A and D have the same eigenvalues. So the eigenvalues of A are 4,6 and 8.

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Similarity

Let A and B be $n \times n$ matrices.

- A is *similar* to B if there is an invertible matrix P such that $P^{-1}AP = B$.
- If A is similar to B, then B is similar to A (because if $P^{-1}AP = B$, and we let $Q = P^{-1}$, then $Q^{-1}BQ = PBP^{-1} = PP^{-1}BP^{-1}P = A$).
- If A is similar to B (and thus B is similar to A), then we say that A and B *are similar*.

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The characteristic polynomials of similar matrices

Theorem 4

If two matrices A and B are similar, then they have the same characteristic polynomial and hence the same eigenvalues (with the same multiplicity).

The *multiplicity* of an eigenvalue λ_k of an $n \times n$ matrix A is the maximal m such that $(\lambda - \lambda_k)^m$ is a factor of $\det(A - \lambda I_n)$.

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Proof of Theorem 4

If A and B are similar, then there is an invertible matrix P such that $P^{-1}AP = B$. In that case,
$B - \lambda I_n = P^{-1}AP - \lambda P^{-1} I_n P = P^{-1}(A - \lambda I_n)P$ and

$$\begin{aligned}
\det(B - \lambda I_n) &= \det(P^{-1}(A - \lambda I_n)P) \\
&= \det(P^{-1}) \det(A - \lambda I_n) \det(P) \\
&= \det(P^{-1}) \det(P) \det(A - \lambda I_n) \\
&= \det(P^{-1}P) \det(A - \lambda I_n) \\
&= \det(I_n) \det(A - \lambda I_n) \\
&= \det(A - \lambda I_n).
\end{aligned}$$

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Finding eigenvalues via the QR algorithm

The previous theorem can be used to calculate the eigenvalues of a square matrix.

- If A is an $n \times n$-matrix, then there exists under certain conditions a sequence $A_1, A_2, \ldots$ of $n \times n$-matrix such that each A_k is similar to A, and such that A_k converges to a triangular matrix B.
- Since each A_k is similar to A, the eigenvalues of each A_k, and hence of B, are the same as the eigenvalues of A.
- So the diagonal entries of A_k converge to the eigenvalues of A.

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Diagonalizable matrices

- A *diagonal* matrix is a square matrix whose entries outside the main diagonal are all zero.
- If D is a diagonal matrix, then the eigenvalues of D are the entries on the main diagonal.
- If D is a diagonal matrix, then $\mathbf{e}_i$ is an eigenvector of D corresponding to the i'th entry on the main diagonal.
- A matrix A is *diagonalizable* if it is similar to a diagonal matrix (that is, if $A = PDP^{-1}$ for some invertible matrix P and some diagonal matrix D).

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The diagonalization Theorem

Theorem 5

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.
In fact, $A = PDP^{-1}$, with D a diagonal matrix, if and only if the columns of P are n linearly independent eigenvectors of A, in which case the diagonal entries of D are eigenvalues of A corresponding, respectively, to the eigenvectors in P.

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Proof of Theorem 5

Suppose that $P = [\mathbf{v}_1\ \mathbf{v}_2 \ldots \mathbf{v}_n]$ is an invertible matrix,

$$D = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}, \text{ and } A = PDP^{-1}.$$

Then

$$A\mathbf{v}_k = PDP^{-1}\mathbf{v}_k = PD\mathbf{e}_k = P\lambda_k\mathbf{e}_k = \lambda_k\mathbf{v}_k,$$

so λ_k is an eigenvalue of A, and $\mathbf{v}_k$ is a corresponding eigenvector. Since P is invertible, the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is linearly independent, so A has n linearly independent eigenvectors.

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Proof of Theorem 5 (cont.)

Conversely, suppose A is an $n \times n$ matrix which has n linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. Let for each k, λ_k be the eigenvalue corresponding to $\mathbf{v}_k$. Let

$$P = [\mathbf{v}_1\ \mathbf{v}_2 \ldots \mathbf{v}_n] \text{ and } D = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}.$$

Then we have for each k that

$$PD\mathbf{e}_k = P\lambda_k\mathbf{e}_k = \lambda_k\mathbf{v}_k$$

and that

$$AP\mathbf{e}_k = A\mathbf{v}_k = \lambda_k\mathbf{v}_k.$$

It follows that $PD = AP$.

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Proof of Theorem 5 (cont.)

Since $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is linearly independent, it follows that P is invertible, so $A = PDP^{-1}$.

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How to diagonalize a matrix (if it is diagonalizable)

1. Find the eigenvalues of A.
2. Find n linearly independent eigenvectors of A (if possible).
3. Construct P from the linearly independent eigenvectors of A.
4. Construct D from the corresponding eigenvalues.
5. Then $A = PDP^{-1}$.

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Example

Let us determine if $A = \begin{bmatrix} 0 & 2 \\ -3 & 5 \end{bmatrix}$ is diagonalizable and diagonalize it if it is and find an expression for A^n.

Solution

The characteristic polynomial of A is

$$\det(A - \lambda I_2) = \begin{vmatrix} -\lambda & 2 \\ -3 & 5-\lambda \end{vmatrix} = -\lambda(5-\lambda) + 6 = \lambda^2 - 5\lambda + 6.$$

The zeros of $\lambda^2 - 5\lambda + 6$ are $\lambda = \frac{5 \pm \sqrt{5^2 - 4 \cdot 6}}{2} = \frac{5 \pm 1}{2}$, so the eigenvalues of A are 2 and 3.

$A - 2I_2 = \begin{bmatrix} -2 & 2 \\ -3 & 3 \end{bmatrix}$, so $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue 2.

$A - 3I_2 = \begin{bmatrix} -3 & 2 \\ -3 & 2 \end{bmatrix}$, so $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue 3.

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Solution (cont.)

Since $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}2\\3\end{bmatrix}$ are two linearly independent eigenvectors of A, it follows that A is diagonalizable.

Let $P = \begin{bmatrix}1 & 2\\1 & 3\end{bmatrix}$ and $D = \begin{bmatrix}2 & 0\\0 & 3\end{bmatrix}$.

Then $AP = \begin{bmatrix}0 & 2\\-3 & 5\end{bmatrix}\begin{bmatrix}1 & 2\\1 & 3\end{bmatrix} = \begin{bmatrix}2 & 6\\2 & 9\end{bmatrix}$

and $PD = \begin{bmatrix}2 & 0\\0 & 3\end{bmatrix}\begin{bmatrix}1 & 2\\1 & 3\end{bmatrix} = \begin{bmatrix}2 & 6\\2 & 9\end{bmatrix}$.

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Solution (cont.)

P is invertible since $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}2\\3\end{bmatrix}$ are linearly independent, so it follows that $A = PDP^{-1}$ and that

$$\begin{aligned} A^n = (PDP^{-1})^n &= \begin{bmatrix}1 & 2\\1 & 3\end{bmatrix}\begin{bmatrix}2^n & 0\\0 & 3^n\end{bmatrix}\begin{bmatrix}3 & -2\\-1 & 1\end{bmatrix} \\ &= \begin{bmatrix}1 & 2\\1 & 3\end{bmatrix}\begin{bmatrix}3\cdot 2^n & -2^{n+1}\\-3^n & 3^n\end{bmatrix} \\ &= \begin{bmatrix}3\cdot 2^n - 2\cdot 3^n & -2^{n+1} + 2\cdot 3^n\\3\cdot 2^n - 3^{n+1} & -2^{n+1} + 3^{n+1}\end{bmatrix}. \end{aligned}$$

Matrices with n distinct eigenvalues are diagonalizable

Theorem 6

An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

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Proof of Theorem 6

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be eigenvectors corresponding to the n distinct eigenvalues. Then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is linearly independent by Theorem 2. So it follows from Theorem 5 that A is diagonalizable.

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Matrices with fewer than n distinct eigenvalues

Theorem 7

Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \ldots, \lambda_p$.

1. For $1 \leq k \leq p$, the dimension of the eigenspace for λ_k is less than or equal to the multiplicity of the eigenvalue λ_k.

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Matrices with fewer than n distinct eigenvalues

Theorem 7

2. A is diagonalizable if and only if the sum of the dimensions of the eigenspaces equals n, and this happens if and only if the characteristic polynomial factors completely into linear factors and the dimension of the eigenspace for each λ_k equals the multiplicity of λ_k.
3. If A is diagonalizable and $\mathcal{B}_k$ is a basis for the eigenspace corresponding to λ_k for each k, then the total collection of vectors in $\mathcal{B}_1, \ldots, \mathcal{B}_k$ forms an eigenvector basis for $\mathbb{R}^n$.

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Problem 7 from the exam from June 2012

1. Find all the eigenvalues of $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & -1 & 4 \end{bmatrix}$.
2. Find a basis for each eigenspace of A. Is A diagonalisable?

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Solution

The characteristic polynomial of A is

$$\begin{aligned}\det(A-\lambda I_3) &= \begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 1-\lambda & 2 \\ 0 & -1 & 4-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 1-\lambda & 2 \\ -1 & 4-\lambda \end{vmatrix} \\ &= (2-\lambda)\big((1-\lambda)(4-\lambda)+2\big) \\ &= (2-\lambda)(\lambda^2-5\lambda+6).\end{aligned}$$

The zeros of $\lambda^2-5\lambda+6$ are $\lambda = \frac{5\pm\sqrt{5^2-4\cdot 6}}{2} = \frac{5\pm 1}{2}$, so the eigenvalues of A are 2 and 3 (notice that 2 has multiplicity 2).

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Solution (cont.)

To find a basis for the eigenspace of A corresponding to the eigenvalue 2, we reduce $A - 2I_3$ to its reduced echelon form.

$$A - 2I_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & -1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We see that the general solution to the equation $(A - 2I_3)\mathbf{x} = \mathbf{0}$ is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ -2x_3 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

where x_1 and x_3 are free parameters.

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Solution (cont.)

It follows that $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \right\}$ is a basis for the eigenspace of A corresponding to the eigenvalue 2.

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Solution (cont.)

To find a basis for the eigenspace of A corresponding to the eigenvalue 3, we reduce $A - 3I_3$ to its reduced echelon form.

$$A - 3I_3 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & -1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

We see that the general solution to the equation $(A - 3I_3)\mathbf{x} = \mathbf{0}$ is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

where x_3 is a free parameter.

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Solution (cont.)

It follows that $\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis for the eigenspace of A corresponding to the eigenvalue 3.

$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ is not in Span $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \right\}$ (because if it was, then it would be an eigenvector of A corresponding to both 2 and 3), so $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is linearly independent. So A has 3 linearly independent eigenvectors, and it is therefore diagonalizable.

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Problem 11 from August 2004

Let A be a diagonalizable matrix where every eigenvalue is either 1 or -1. Show that $A^{-1} = A$.

Solution

If A be a diagonalizable matrix where every eigenvalue is either 1 or -1, then $A = P^{-1}DP$ where P is an invertible matrix and D is a diagonal matrix where all the entries on the diagonal is either 1 or -1.
$D^2 = I$, so D is invertible and $D^{-1} = D$. It follows that A is invertible and that
$A^{-1} = (P^{-1}DP)^{-1} = P^{-1}D^{-1}(P^{-1})^{-1} = P^{-1}DP = A$.

Problem 7 from December 2005

Let A and B be two $n \times n$-matrices.

1. Show that if 0 is an eigenvalue of AB, then 0 is an eigenvalue of BA.
2. Show that if B is invertible and λ is an eigenvalue of AB, then λ is an eigenvalue of BA.

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Solution

Since the eigenvalues of an $n \times n$-matrix C are the zeros of the characteristic polynomial $\det(C - \lambda I_n)$, it follows that 0 is an eigenvalue of C if and only if $\det(C) = 0$. So if 0 is an eigenvalue of AB, then $\det(BA) = \det(AB) = 0$, and then 0 is an eigenvalue of BA.

Suppose now that λ is a nonzero eigenvalue of AB, and let $\mathbf{x}$ be a corresponding eigenvector. Then $AB\mathbf{x} = \lambda\mathbf{x}$, so if we let $\mathbf{z} = B\mathbf{x}$, then $BA\mathbf{z} = BAB\mathbf{x} = B\lambda\mathbf{x} = \lambda B\mathbf{x} = \lambda\mathbf{z}$. The vector $\mathbf{z}$ must be nonzero, because otherwise $\lambda\mathbf{x} = AB\mathbf{x} = A\mathbf{z} = 0$ which would contradict the fact that $\lambda \neq 0$ and $\mathbf{x} \neq \mathbf{0}$.

So $\mathbf{z}$ is an eigenvector of BA, and λ is the corresponding eigenvalue.

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Review quiz

Let A be a square matrix. Decide for each of the following statements, whether the statement is true or false.

1. The sum of two eigenvectors of a matrix A is also an eigenvector of A.
2. Two eigenvectors corresponding to the same eigenvalue are always linearly dependent.
3. If A and B are invertible $n \times n$ matrices, then AB is similar to BA.
4. Each eigenvector of A is also an eigenvector of A^2.
5. If A is an $n \times n$ diagonalizable matrix, then each vector in $\mathbb{R}^n$ can be written as a linear combination of eigenvectors of A.
6. If A contains a row or column of zeros, then 0 is an eigenvalue of A.

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Solution

1. False, if $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of A corresponding to different eigenvalues, then $\mathbf{v}_1 + \mathbf{v}_2$ cannot be an eigenvector of A.
2. False, if an eigenspace has dimension higher than 1, then it contains two linearly independent eigenvectors.
3. True, because $AB = A(BA)A^{-1}$.
4. True, if $A\mathbf{x} = \lambda\mathbf{x}$, then $A^2\mathbf{x} = A\lambda\mathbf{x} = \lambda^2\mathbf{x}$.
5. True, if A is an $n \times n$ diagonalizable matrix, then it has n linearly independent eigenvectors, and these eigenvectors will then form a basis of $\mathbb{R}^n$.
6. True, because 0 is an eigenvalue of A if and only if $\det(A) = 0$, and if A contains a row or column of zeros, then $\det(A) = 0$.

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Plan for next week

Next week we shall study

- real matrices with complex eigenvalues,
- systems of first order differential equations.

Sections 5.5 and 5.7 in “Linear Algebras and Its Applications” (pages 295-301, and 311-319), and Section 4.2 in “Second-Order Equations” (pages xlv-xlix).

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