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TMA4115 - Calculus 3

Lecture 21, April 3

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Review of last week's lecture

Last week we introduce and studied

- the *dimension* of a vector space,
- the *rank* of a matrix,
- *Markov chains*.

Today's lecture

Today we shall introduce and study

- *eigenvectors*, *eigenvalues* and *eigenspaces* of square matrices,
- the *characteristic polynomial* of a square matrix.

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Eigenvectors and eigenvalues of square matrices

Let A be an $n \times n$ matrix.

- An *eigenvector* of A is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ.
- An *eigenvalue* of A is a scalar λ such that the equation $A\mathbf{x} = \lambda\mathbf{x}$ has a nontrivial solution.
- If λ is an eigenvalue and $\mathbf{x}$ is an eigenvector such that $A\mathbf{x} = \lambda\mathbf{x}$, then $\mathbf{x}$ is called an *eigenvector (of A) corresponding to* λ and λ is called the *eigenvalue (of A) corresponding to* $\mathbf{x}$.

Example

Consider the matrix $A = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}$.

$$A \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

so 2 is an eigenvalue of A, and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector of A corresponding to the eigenvalue 2.

$$A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix} = -1 \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

so -1 is an eigenvalue of A, and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector of A corresponding to the eigenvalue -1.

Eigenspaces

Let A be an $n \times n$ matrix and let λ be an eigenvalue of A.

- A vector $\mathbf{x} \in \mathbb{R}^n$ is an eigenvector of A corresponding to λ if and only if x is a nontrivial solution of the equation $(A - \lambda I_n)\mathbf{x} = \mathbf{0}$.
- The set of of all solution of the equation $(A - \lambda I_n)\mathbf{x} = \mathbf{0}$ (i.e., $\text{Nul}(A - \lambda I_n)$) is called the *eigenspace (of A) corresponding to* λ.
- Notice that an eigenspace of A is a subspace of $\mathbb{R}^n$.
- Notice also that $\mathbf{0}$ belongs to any eigenspace, even though it is not an eigenvector.

Example

Consider again the matrix $A = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}$. We have seen that 2 and -1 are eigenvalues of A. Let us find the corresponding eigenspaces.

The equation $A\mathbf{x} = 2\mathbf{x}$ is equivalent to the equation $(A - 2I_2)\mathbf{x} = \mathbf{0}$. To solve the latter equation, we reduce the matrix $(A - 2I_2)$ its reduced echelon form.

$$\begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}.$$

We see that the general solution to the equation $(A - 2I_2)\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ where x_2 is a free parameter.

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Example (cont.)

It follows that the eigenspace of A corresponding to the eigenvalue 2 is Span $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$.

Similar, the equation $A\mathbf{x} = -1\mathbf{x}$ is equivalent to the equation $(A + I_2)\mathbf{x} = \mathbf{0}$. To solve the latter equation, we reduce the matrix $(A + I_2)$ its reduced echelon form.

$$\begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1/2 \\ 2 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix}.$$

We see that the general solution to the equation $(A + I_2)\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$ where x_2 is a free parameter.

Example (cont.)

It follows that the eigenspace of A corresponding to the eigenvalue -1 is Span $\left\{ \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \right\}$.

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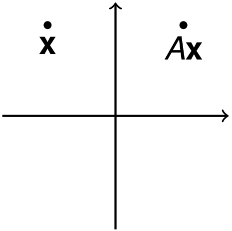
Example

Let $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. Then the linear transformation $\mathbf{x} \to A\mathbf{x}$ sends a vector $\mathbf{x}$ to the vector which corresponds to the point we get by reflecting the point corresponding to $\mathbf{x}$ in the x_2-axis.

It follows that -1 is an eigenvalue of A and that the corresponding eigenspace is the x_1-axis, and that 1 is an eigenvalue of A and that the corresponding eigenspace is the x_2-axis.



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Linearly independent eigenvectors

Theorem 2

If $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is linearly independent.

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Proof of Theorem 2

Assume, for contradiction, that $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is linearly dependent. Then there is a $1 \leq p < r$ and such that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is linearly independent and $\mathbf{v}_{p+1} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$ for some scalars $c_1, c_2, \ldots, c_p$. Then

$$\lambda_{p+1}\mathbf{v}_{p+1} = A\mathbf{v}_{p+1} = A(c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p) = c_1\lambda_1\mathbf{v}_1 + \cdots + c_p\lambda_p\mathbf{v}_p$$

and

$$\lambda_{p+1}\mathbf{v}_{p+1} = \lambda_{p+1}(c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p) = c_1\lambda_{p+1}\mathbf{v}_1 + \cdots + c_p\lambda_{p+1}\mathbf{v}_p$$

so

$$\mathbf{0} = c_1(\lambda_1 - \lambda_{p+1})\mathbf{v}_1 + \cdots + c_p(\lambda_p - \lambda_{p+1})\mathbf{v}_p.$$

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Proof of Theorem 2

Since $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ is linearly independent and $\lambda_k - \lambda_{p+1} \neq 0$ for each $k = 1, \dots, p$, it follows that $c_1 = c_2 = \cdots = c_p = 0$. But then $\mathbf{v}_{p+1} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$ which contradicts the fact that $\mathbf{v}_{p+1}$ is an eigenvector. So it most be the case that $\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$ is linearly independent.

Finding a basis of an eigenspace

Let A be an $n \times n$ matrix and let λ be an eigenvalue of A. The eigenspace of A corresponding to λ is $\text{Nul}(A - \lambda I_n)$. So we can find a basis for the eigenspace of A corresponding to λ by:

1. row reducing $A - \lambda I_n$ to its reduced echelon form,
2. write the solutions to the equation $(A - \lambda I_n)\mathbf{x} = \mathbf{0}$ (which is equivalent to the equation $A\mathbf{x} - \mathbf{0}$) as a linear combinations of vectors using the free variables as parameters.
3. Then these vectors form a basis of the eigenspace of A corresponding to λ.

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Example

Let $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$.

Let us determine if 2 is an eigenvalue of A, and let us find a basis for the corresponding eigenspace if it is.

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Solution

2 is an eigenvalue of A if and only if the equation $(A - 2I_3)\mathbf{x} = \mathbf{0}$ has a nontrivial solution in which case the solution set of the equation is the eigenspace of A corresponding to 2. We find the solution set of the equation $(A - 2I_3)\mathbf{x} = \mathbf{0}$ by reducing $A - 2I_3$ to its reduced echelon form.

$$A - 2I_3 = \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix} \to \begin{bmatrix} 2 & -1 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1/2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Solution (cont.)

We see that the general solution to the equation $(A - 2I_3)\mathbf{x} = \mathbf{0}$ is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x_2 - 3x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$ where x_2 and x_3 are free parameters.

It follows that 2 is an eigenvalue of A, and that

$$\left\{ \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$ is a basis for eigenspace of A corresponding to the eigenvalue 2.

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Finding the eigenvalues of a matrix

Let A be an $n \times n$ matrix.

- A scalar λ is an eigenvalue of A if and only if $\det(A - \lambda I_n) = 0$.
- The equation $\det(A - \lambda I_n) = 0$ is called the *characteristic equation* of A.
- If we regard λ as an independent variable, then $\det(A - \lambda I_n)$ is a polynomial of degree n in λ.
- $\det(A - \lambda I_n)$ is called the *characteristic polynomial* of A.
- The eigenvalues of A are the zeros of the characteristic polynomial $\det(A - \lambda I_n)$ of A.

Example

Let us find the eigenvalues of the matrix $A = \begin{bmatrix} 3 & 0 \\ 4 & -2 \end{bmatrix}$.

The characteristic polynomial of A is

$$\det(A - \lambda I_2) = \begin{vmatrix} 3-\lambda & 0 \\ 4 & -2-\lambda \end{vmatrix} = (3-\lambda)(-2-\lambda) = \lambda^2 - \lambda - 6.$$

The solutions of the characteristic equation $\lambda^2 - \lambda - 6 = 0$ are 3 and -2. So the eigenvalues of A are 3 and -2.

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Example

In a certain region, 5% of a city's population moves to the surrounding suburbs each year, and 3% of the suburban population moves into the city. In 2000, there were 600,000 residents in the city and 400,000 in the suburbs.
Let us try to find a formula for the number of the people in the city and the number of people in the suburbs for each year.

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Solution (cont.)

Let $\mathbf{x} = \begin{bmatrix} \text{the population in the city in year } 2000+i \\ \text{the population in the suburbs in year } 2000+i \end{bmatrix}$

and $A = \begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix}$. Then $\mathbf{x}_{i+1} = A\mathbf{x}_i$ for all i. It follows that $\mathbf{x}_i = A^i\mathbf{x}_0$ for all i.

The characteristic polynomial of A is

$$\begin{aligned} \det(A - \lambda I_2) &= \begin{vmatrix} 0.95-\lambda & 0.03 \\ 0.05 & 0.97-\lambda \end{vmatrix} \\ &= (0.95-\lambda)(0.97-\lambda) - 0.0015 \\ &= \lambda^2 - 1.92\lambda + 0.92. \end{aligned}$$

The solutions of the characteristic equation $\lambda^2 - 1.92\lambda + 0.92 = 0$ are 1 and 0.92.

Solution (cont.)

It follows that the eigenvalues of A are 1 and 0.92.

The matrix $A - I_2 = \begin{bmatrix} -0.05 & 0.03 \\ 0.05 & -0.03 \end{bmatrix}$ is row equivalent to the matrix $\begin{bmatrix} -5 & 3 \\ 0 & 0 \end{bmatrix}$. It follows that $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$ is an eigenvector of A corresponding to the eigenvalue 1.

The matrix $A - 0.92I_2 = \begin{bmatrix} 0.03 & 0.03 \\ 0.05 & 50.05 \end{bmatrix}$ is row equivalent to the matrix $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$. It follows that $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector of A corresponding to the eigenvalue 0.92.

Solution (cont.)

$\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^2$, so there exists scalars c_1 and c_2 such that $\mathbf{x}_0 = c_1 \begin{bmatrix} 3 \\ 5 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ is the coordinate vector of $\mathbf{x}_0$ with respect to $\mathcal{B}$, so

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix}^{-1} \mathbf{x}_0 = \frac{1}{-8} \begin{bmatrix} -1 & -1 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 600,000 \\ 400,000 \end{bmatrix} = \begin{bmatrix} 125,000 \\ 225,000 \end{bmatrix}$$

and $\mathbf{x}_0 = 125,000 \begin{bmatrix} 3 \\ 5 \end{bmatrix} + 225,000 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

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Solution (cont.)

It follows that

$$\mathbf{x}_i = A^i\mathbf{x}_0 = A^i\left(125,000\begin{bmatrix}3\\5\end{bmatrix} + 225,000\begin{bmatrix}1\\-1\end{bmatrix}\right) =$$

$$125,000\begin{bmatrix}3\\5\end{bmatrix} + 225,000 \cdot (0.92)^i\begin{bmatrix}1\\-1\end{bmatrix} \text{ for all } i.$$

Since $(0.92)^i \to 0$ as $i \to \infty$, it follows that

$$\mathbf{x}_i \to 125,000\begin{bmatrix}3\\5\end{bmatrix} = \begin{bmatrix}375,000\\625,000\end{bmatrix} \text{ as } i \to \infty.$$

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Markov chains

- A *probability vector* is a vector with nonnegative entries that add up to 1.
- A *stochastic matrix* is a square matrix whose columns are probability vectors.
- A *Markov chain* is a sequence $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots$ of probability vectors together with a stochastic matrix P such that $\mathbf{x}_1 = P\mathbf{x}_0$, $\mathbf{x}_2 = P\mathbf{x}_1$, $\mathbf{x}_3 = P\mathbf{x}_2$, $\ldots$.
- A *steady-state vector* (or an *equilibrium vector*) for a stochastic matrix P is a probability vector $\mathbf{q}$ such that $P\mathbf{q} = \mathbf{q}$.
- So a steady-state vector for a stochastic matrix P is a probability vector which is an eigenvector of P corresponding to the eigenvalue 1.

Problem 7 from the exam from August 2010

Let $A = \begin{bmatrix} 2 & 0 & 2 \\ 1 & -2 & -1 \\ -1 & 6 & 5 \end{bmatrix}$.

1. Solve the equation $A\mathbf{x} = \mathbf{0}$.
2. Find the eigenvalues and eigenvectors of A.

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Solution

To solve the equation $A\mathbf{x} = \mathbf{0}$, we reduce A to its reduced echelon form.

$$\begin{bmatrix} 2 & 0 & 2 \\ 1 & -2 & -1 \\ -1 & 6 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 1 & -2 & -1 \\ -1 & 6 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & -2 & -2 \\ 0 & 6 & 6 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 6 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

We see that the general solution to the equation $A\mathbf{x} = \mathbf{0}$ is

$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ where x_3 is a free parameters.

Solution (cont.)

The characteristic polynomial of A is

$$\begin{aligned}\det(A-\lambda I_3) &= \begin{vmatrix} 2-\lambda & 0 & 2 \\ 1 & -2-\lambda & -1 \\ -1 & 6 & 5-\lambda \end{vmatrix} \\ &= (2-\lambda)\begin{vmatrix} -2-\lambda & -1 \\ 6 & 5-\lambda \end{vmatrix} + 2\begin{vmatrix} 1 & -2-\lambda \\ -1 & 6 \end{vmatrix} \\ &= (2-\lambda)\big((-2-\lambda)(5-\lambda)+6\big) + 2\big(6+(-2-\lambda)\big) \\ &= -\lambda^3+5\lambda^2-4\lambda = -\lambda(\lambda^2-5\lambda+4).\end{aligned}$$

The zeros of $\lambda^2-5\lambda+4$ are 1 and 4, so the eigenvalues of A are 0,1 and 4.

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Solution (cont.)

It follows from the first part that the eigenvectors of A corresponding to the eigenvalue of 0 are the vectors

$$\left\{ t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} : t \neq 0 \right\}.$$

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Solution (cont.)

To find the eigenvectors of A corresponding to the eigenvalue 4, we solve the equation $(A - 4I_3)\mathbf{x} = \mathbf{0}$. To solve the equation $(A - 4I_3)\mathbf{x} = \mathbf{0}$, we reduce $A - 4I_3$ to its reduced echelon form.

$$A - 4I_3 = \begin{bmatrix} -2 & 0 & 2 \\ 1 & -6 & -1 \\ -1 & 6 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 1 & -6 & -1 \\ -1 & 6 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & -6 & 0 \\ 0 & 6 & 0 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & -6 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

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Solution (cont.)

We see that the solution to the equation $(A - 4I_3)\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ where x_3 is a free parameters. It follows that the eigenvectors of A corresponding to the eigenvalue of 4 are the vectors $\left\{ t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} : t \neq 0 \right\}$.

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Solution (cont.)

To find the eigenvectors of A corresponding to the eigenvalue 1, we solve the equation $(A - I_3)\mathbf{x} = \mathbf{0}$. To solve the equation $(A - I_3)\mathbf{x} = \mathbf{0}$, we reduce $A - I_3$ to its reduced echelon form.

$$A - I_3 = \begin{bmatrix} 1 & 0 & 2 \\ 1 & -3 & -1 \\ -1 & 6 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & -3 & -3 \\ 0 & 6 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution (cont.)

We see that the solution to the equation $(A - I_3)\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$ where x_3 is a free parameters. It follows that the eigenvectors of A corresponding to the eigenvalue of 1 are the vectors $\left\{ t \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} : t \neq 0 \right\}$.

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Problem 6 from August 2012

For which numbers a does $\mathbb{R}^2$ have a basis of eigenvectors of the matrix $\begin{bmatrix} 0 & a \\ 1 & 0 \end{bmatrix}$?

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Solution

Let $A = \begin{bmatrix} 0 & a \\ 1 & 0 \end{bmatrix}$. The characteristic polynomial of A is

$$\det(A - \lambda I_2) = \begin{vmatrix} -\lambda & a \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - a.$$

If $a > 0$, then A has two distinct eigenvalues $\pm\sqrt{a}$. If $\mathbf{v}_1$ is an eigenvector corresponding to $\sqrt{a}$, and $\mathbf{v}_2$ is an eigenvector corresponding to $-\sqrt{a}$, then $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent, and therefore a basis of $\mathbb{R}^2$.

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Solution (cont.)

If $a = 0$, then A has one eigenvalue 0. The corresponding eigenspace is $\mathrm{Nul}(A) = \mathrm{Span}\left\{\begin{bmatrix}0\\1\end{bmatrix}\right\}$ which is one-dimensional. It follows that all eigenvectors of A are linearly dependent, and thus that $\mathbb{R}^2$ does not have a basis of eigenvectors of A is this case.

If $a < 0$, then A does not have any real eigenvalues, and therefore $\mathbb{R}^2$ does not have a basis of eigenvectors of A is this case (A has complex eigenvalues, but the corresponding eigenvectors will not belong to $\mathbb{R}^2$ and therefore no eigenvectors of A will form a basis for $\mathbb{R}^2$)..

So $\mathbb{R}^2$ have a basis of eigenvectors of A if and only if $a > 0$.

Tomorrow's lecture

Tomorrow we shall look at

- *diagonal* matrices,
- how to *diagonalizable* a matrix.

Section 5.3 in "Linear Algebras and Its Applications" (pages 281—288).