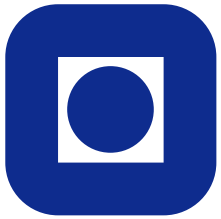

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TMA4115 - Calculus 3

Lecture 20, March 21

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Yesterday's lecture

Yesterday we introduced and studied

- the *dimension* of a vector space,
- the *rank* of a matrix.

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Today's lecture

Today we shall look at

- the *rank* of a matrix,
- *Markov chains*.

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The dimension of a vector space

Theorem 10

If a vector space has a basic of n vectors, then every basis of V must consist of exactly n vectors.

- If V is spanned by a finite set, then V is said to be *finite-dimensional*.
- If V is a finite dimensional vector space, then the *dimension* of V, written as $\dim(V)$, is the number of vectors in a basis for V.
- The dimension of the zero vector space $\{\mathbf{0}\}$ is defined to be zero.
- If V is not spanned by a finite set, then V is said to be *infinite-dimensional*.

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Why is the dimension of a vector space useful?

If V is a vector space of dimension n, then the coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is an *isomorphism* between V to $\mathbb{R}^n$ for any basis $\mathcal{B}$ of V.

Theorem 12

Let V be a p-dimensional vector space, where $p \geq 1$.

1. Any linearly independent set of exactly p elements in V is automatically a basis for V
2. Any set of exactly p elements that spans V is automatically a basis for V.

It follows that if V is a finite dimensional vector space, H is a subspace of V and $\dim(H) = \dim(V)$, then $H = V$.

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Example

The Chebyshev polynomials of the first kind is the unique sequence of polynomials $T_0, T_1, T_2, \ldots$ satisfying $\mathbb{T}_n(\cos\theta) = \cos(n\theta)$ for all θ.
They satisfy the recurrence relation

$$T_0(x) = 1, \;\; T_1(x) = x, \;\; T_{n+1} = 2xT_n(x) - T_{n-1}(x).$$

It follows that T_n has degree n. We therefore have that $\{T_0, T_1, T_2, \ldots\}$ is linearly independent because T_0 is not the zero polynomial and T_n is not a linear combination of $T_0, T_1, \ldots, T_{n-1}$. It follows that $\{T_0, T_1, \ldots, T_n\}$ is a basis of $\mathbb{P}_n$. So every polynomial can be written as a unique linear combination of $T_0, T_1, T_2, \ldots$.

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The dimension of the column space of a matrix

Recall that if $A = [\mathbf{a}_1\ \mathbf{a}_2 \ldots \mathbf{a}_n]$ is an $m \times n$ matrix, then $\text{Col}(A) = \text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$.
Recall also that the pivot columns of A form a basis for $\text{Col}(A)$.
It follows that the dimension of $\text{Col}(A)$ is the number of pivot positions in A.

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The dimension of the null space of a matrix

Recall that if A is an $m \times n$ matrix, then
$\text{Nul}(A) = \{\mathbf{x} : \mathbf{x} \text{ is in } \mathbb{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\}$.
Recall also that we can find a basis for $\text{Nul}(A)$ by

1. row reducing A to its reduced echelon form,
2. writing the solutions to the equation $A\mathbf{x} = \mathbf{0}$ as a linear combinations of vectors using the free variables as parameters,
3. forming a basis for $\text{Nul}(A)$ by taken the set of the vectors we use to write the solutions to the equation $A\mathbf{x} = \mathbf{0}$.

It follows that the dimension of $\text{Nul}(A)$ is the number of nonpivot columns of A.

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Problem 5 from June 2012

Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be defined by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x + y + z \\ -x + 3y + z \\ 2x - z \\ y + 4z \end{bmatrix}$.

a) Find a matrix A such that $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = A\begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

b) Find $\dim(\text{Nul}(A))$ and a basis for $\text{Col}(A)$. Is T one-to-one (injective)? Is T onto (surjective)?

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Solution

If $A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 3 & 1 \\ 2 & 0 & -1 \\ 0 & 1 & 4 \end{bmatrix}$, then

$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x + y + z \\ -x + 3y + z \\ 2x - z \\ y + 4z \end{bmatrix} = T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right).$$

Solution (cont.)

Next, we reduce A to an echelon form

$$A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 3 & 1 \\ 2 & 0 & -1 \\ 0 & 1 & 4 \end{bmatrix} \to \begin{bmatrix} -1 & 3 & 1 \\ 2 & 1 & 1 \\ 2 & 0 & -1 \\ 0 & 1 & 4 \end{bmatrix} \to \begin{bmatrix} -1 & 3 & 1 \\ 0 & 7 & 3 \\ 0 & 6 & 1 \\ 0 & 1 & 4 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -3 & -1 \\ 0 & 7 & 3 \\ 0 & 6 & 1 \\ 0 & 1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & -1 \\ 0 & 1 & 4 \\ 0 & 7 & 3 \\ 0 & 6 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & -1 \\ 0 & 1 & 4 \\ 0 & 0 & -25 \\ 0 & 0 & -23 \end{bmatrix}$$

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Solution (cont.)

We see that A has a pivot position in every column. It follows that $\operatorname{rank}(A) = 3$ and that $\left\{ \begin{bmatrix} 2 \\ -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ 4 \end{bmatrix} \right\}$ is a basis of $\operatorname{Col}(A)$.

It follows that $\dim(\operatorname{Nul}(A)) = 3 - \operatorname{rank}(A) = 0$. Since $\dim(\operatorname{Nul}(A)) = 0$, T is one-to-one (because T is one-to-one $\iff \ker(T) = \{0\} \iff \operatorname{Nul}(A) = \{0\} \iff \dim(\operatorname{Nul}(A)) = 0$).

Since $\dim(\operatorname{Col}(A)) = 3 < 4$, T is not onto (because T is onto $\iff T(\mathbb{R}^3) = \mathbb{R}^4 \iff \operatorname{Col}(A) = \mathbb{R}^4 \iff \dim(\operatorname{Col}(A)) = 4$).

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The dimension of the row space of a matrix

Recall that the *row space* of a matrix A is set of all linear combination of the row vectors of A. Recall also that we have the following theorem:

Theorem 13

Let A and B be matrices.

1. If A and B are row equivalent, then $\text{Row}(A) = \text{Row}(B)$.
2. If B is in echelon form, then the nonzero rows of B form a basis for $\text{Row}(B)$.

It follows that the dimension of $\text{Row}(A)$ is the number of pivot positions in A.

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The rank of a matrix

Let A be a matrix. Then the *rank* of A, written rank(A), is the dimension of the column space Col(A) of A.

Theorem 14

Let A be an $m \times n$ matrix. Then the following numbers are equal:

- rank(A)
- dim(Col(A))
- dim(Row(A))
- $n -$ dim(Nul(A))
- The number of pivot positions in A.

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Example

Let A be a nonzero 3×3 matrix. Suppose that

$$A \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} = A \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \mathbf{0}.$$

What is $\dim(\text{Row}(A))$?

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Solution

Since $\begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ are linearly independent and in $\text{Nul}(A)$, $\dim(\text{Nul}(A)) \geq 2$.

$\dim(\text{Row}(A)) = \text{rank}(A) = 3 - \dim(\text{Nul}(A))$ so $\dim(\text{Row}(A))$ is either 0 or 1.

Since $A \neq 0$, $\text{Row}(A) \neq \{\mathbf{0}\}$, so $\dim(\text{Row}(A)) > 0$. It follows that $\dim(\text{Row}(A)) = 1$.

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The rank of a matrix and solutions to linear system

Consider a linear system in m variables, and let A be the coefficient matrix of the system and B the augmented matrix of the system.

1. Then $\text{rank}(A) \leq \text{rank}(B)$ (because A is a submatrix of B).
2. The system is consistent if and only if $\text{rank}(A) = \text{rank}(B)$ (because the system is consistent $\iff$ the last column of B is not a pivot column $\iff$ $\text{rank}(A) = \text{rank}(B)$).
3. If the system is consistent, then it has a unique solution if and only if $\text{rank}(A) = m$ (because the system has a unique solution $\iff$ A has a pivot position in every row $\iff$ $\text{rank}(A) = m$).

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The rank of a matrix and solutions to linear system

4. If the system is consistent, then its general solution has $m - \text{rank}(A)$ free variables (because each nonpivot column correspond to a free variable).

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Problem 4 from the exam from August 2011

Let $A = \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ -2 & 3 & -3 & -3 & -4 \\ 4 & -6 & 9 & 5 & 9 \\ -2 & 3 & 3 & -4 & 1 \end{bmatrix}$.

1. Find a basis for the null space, Nul(A), and a basis for the row space, Row(A).
2. Find a basis for the column space, Col(A). What is rank(A).

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Solution

We reduce A to its reduced echelon form.

$$\begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ -2 & 3 & -3 & -3 & -4 \\ 4 & -6 & 9 & 5 & 9 \\ -2 & 3 & 3 & -4 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & -3 & 1 & -1 \\ 0 & 0 & 9 & -2 & 6 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -3 & 6 & 0 & -1 \\ 0 & 0 & 3 & 0 & 4 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow$$

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Solution

$$\begin{bmatrix} 2 & -3 & 0 & 0 & -9 \\ 0 & 0 & 3 & 0 & 4 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3/2 & 0 & 0 & -9/2 \\ 0 & 0 & 1 & 0 & 4/3 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

A basis for Row(A) is $\left\{ \begin{bmatrix} 1 \\ -3/2 \\ 0 \\ 0 \\ -9/2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 4/3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 3 \end{bmatrix} \right\}$.

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Solution

The general solution to the equation $A\mathbf{x} = \mathbf{0}$ is

$$\mathbf{x} = \begin{bmatrix} 3/2x_2 + 9/2x_5 \\ x_2 \\ -4/3x_5 \\ -3x_5 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 3/2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 9/2 \\ 0 \\ -4/3 \\ -3 \\ 1 \end{bmatrix}$$

so $\left\{ \begin{bmatrix} 3/2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 9/2 \\ 0 \\ -4/3 \\ -3 \\ 1 \end{bmatrix} \right\}$ is a basis for Nul(A).

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Solution

A basis for Col(A) is $\left\{ \begin{bmatrix} 2 \\ -2 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \\ 9 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 5 \\ 4 \end{bmatrix} \right\}$.

We see that rank(A) $= 3$.

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The invertible matrix theorem

Theorem

Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is invertible.

1. The columns of A form a basis of $\mathbb{R}^n$.
2. $\text{Col}(A) = \mathbb{R}^n$.
3. $\dim(\text{Col}(A)) = n$.
4. $\text{rank}(A) = n$.
5. $\text{Nul}(A) = \{\mathbf{0}\}$.
6. $\dim(\text{Nul}(A)) = 0$.
7. $\dim(\text{Row}(A)) = n$.
8. The rows of A form a basis of $\mathbb{R}^n$.
9. $\text{Row}(A) = \mathbb{R}^n$.

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Proof

A is invertible $\iff$ the columns of A span $\mathbb{R}^n$
$\iff$ $\text{Col}(A) = \mathbb{R}^n$ $\iff$ $\dim(\text{Col}(A)) = n$ $\iff$ $\text{rank}(A) = n$
$\iff$ $\dim(\text{Nul}(A)) = 0$ $\iff$ $\text{Nul}(A) = \{\mathbf{0}\}$.

A is invertible $\iff$ the columns of A span $\mathbb{R}^n$ and the columns of A are linearly independent $\iff$ the columns of A form a basis of $\mathbb{R}^n$.

A is invertible $\iff$ A^T is invertible
$\iff$ $\text{Row}(A) = \text{Col}(A^T) = \mathbb{R}^n$ $\iff$ $\dim(\text{Row}(A)) = n$.

A is invertible $\iff$ A^T is invertible $\iff$ the row of A (which are the columns of A^T) span $\mathbb{R}^n$ and the rows of A are linearly independent $\iff$ the rows of A form a basis of $\mathbb{R}^n$.

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Example

In a certain region, 5% of a city's population moves to the surrounding suburbs each year, and 3% of the suburban population moves into the city. In 2000, there were 600,000 residents in the city and 400,000 in the suburbs. Let us look at how the populations of the city and of the surrounding suburbs change through the years.

Solution

Let $A = \begin{bmatrix} 95\% & 3\% \\ 5\% & 97\% \end{bmatrix}$ and

$$\mathbf{x}_i = \begin{bmatrix} \text{the proportion of the total population that lives} \\ \text{in the city in year } 2000 + i \\ \text{the proportion of the total population that lives} \\ \text{in the suburbs in year } 2000 + i \end{bmatrix}.$$

Then $\mathbf{x}_{i+1} = A\mathbf{x}_i$, so $\mathbf{x}_i = A^i\mathbf{x}_0$.

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Markov chains

- A *probability vector* is a vector with nonnegative entries that add up to 1.
- A *stochastic matrix* is a square matrix whose columns are probability vectors.
- A *Markov chain* is a sequence $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots$ of probability vectors together with a stochastic matrix P such that $\mathbf{x}_1 = P\mathbf{x}_0$, $\mathbf{x}_2 = P\mathbf{x}_1$, $\mathbf{x}_3 = P\mathbf{x}_2$, ...
- A *steady-state vector* (or an *equilibrium vector*) for a stochastic matrix P is a probability vector $\mathbf{q}$ such that $P\mathbf{q} = \mathbf{q}$.

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Convergence of regular Markov chains

A stochastic matrix P is said to be *regular* if P^k contains only strict positive entries for some k.

Theorem 18

- If P is a regular stochastic matrix, then P has a unique steady-state vector q.
- If $\mathbf{x}_0$ is any initial state and $\mathbf{x}_{k+1} = P\mathbf{x}_k$ for $k = 0, 1, 2, \ldots,$ then the Markov chain $\{\mathbf{x}_k\}_{k \in \mathbb{N}_0}$ converges to $\mathbf{q}$ as $k \to \infty$.

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Applications of Markov chains

Markov chains is a useful tool that can be used to model discrete dynamical systems.
They are for example used in

- physics,
- chemistry,
- information theory,
- statistics,
- economics and finance,
- biology,
- social sciences.

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Example

A very simple weather model says that if it is raining one day, then there is a 60% chance that will also rain the next day, and that if it not raining, then there is a 30% chance that will rain the next day. According to this model, what is the probability that it will rain on a given day?

Solution

Let $P = \begin{bmatrix} 60\% & 30\% \\ 40\% & 70\% \end{bmatrix}$ and

$\mathbf{x}_i = \begin{bmatrix} \text{the probability that it rains on day } i \\ \text{the probability that it does not rain on day } i \end{bmatrix}$.

Then $\mathbf{x}_{i+1} = P\mathbf{x}_i$. P is a regular stochastic matrix, so $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots$ is a regular Markov chain.

Let $\mathbf{q}$ be the steady-state vector of P. Then $P\mathbf{q} = \mathbf{q}$, so $\mathbf{x} = \mathbf{q}$ is a solution to the equation $(P - I_2)\mathbf{x} = \mathbf{0}$. We reduce $P - I_2$ to its reduced echelon form.

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Solution

$$\begin{bmatrix} -4/10 & 3/10 \\ 4/10 & -3/10 \end{bmatrix} \to \begin{bmatrix} -4 & 3 \\ 4 & -3 \end{bmatrix} \to \begin{bmatrix} -4 & 3 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -3/4 \\ 0 & 0 \end{bmatrix}$$

The general solution to the equation $(P - I_2)\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3/4x_2 \\ x_2 \end{bmatrix}$.

So $\mathbf{q} = \begin{bmatrix} 3/4x_2 \\ x_2 \end{bmatrix}$ for some x_2. In order for $\mathbf{q}$ to be a probability vector, we must have that $1 = 3/4x_2 + x_2 = 7/4x_2$, so $x_2 = 4/7$ and $\mathbf{q} = \begin{bmatrix} 3/7 \\ 4/7 \end{bmatrix}$.

Thus, according to the model, the probability that it will rain on a given day is $3/7$.

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Review quiz

Let V be a nonzero finite-dimensional vector space, and let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be a subset of V. Decide for each of the following statements, whether the statement is true or false.

1. The set of all linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ is a vector space.
2. If $\{\mathbf{v}_1, \ldots, \mathbf{v}_{p-1}\}$ spans V, then $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ spans V.
3. Row operations on a matrix can change the null space.
4. The nonzero rows of a matrix A form a basis for Row(A).
5. If H is a subspace of $\mathbb{R}^3$, then there is a 3×3 matrix A such that $H = \text{Col}(A)$.
6. If A is an $m \times n$ matrix and rank $A = m$, then the linear transformation $\mathbf{x} \to A\mathbf{x}$ is one-to-one.

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Solution

1. True (the set of all linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ is $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ which is a subspace of V and therefore a vector space).
2. True ($\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_{p-1}\} \subseteq \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\} \subseteq V$, so if $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_{p-1}\} = V$, then $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\} = V$).
3. False (according to Theorem 13 $\text{Row}(A) = \text{Row}(B)$ if A and B are row equivalent).
4. False (the nonzero rows of an echelon form A form a basis for $\text{Row}(A)$).

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Solution (cont.)

5. True (if H is a subspace of $\mathbb{R}^3$, then $\dim H \leq 3$ so H has a basis consisting of 3 or fewer vectors from $\mathbb{R}^3$, and if A is a 3×3 matrix whose columns are these vectors (if there are fewer vectors than 3, then just repeat some of them), then $H = \text{Col}(A)$).
6. False ($\mathbf{x} \to A\mathbf{x}$ is one-to-one $\iff$ $\text{Nul}(A) = \{0\}$ $\iff$ $\dim(\text{Nul}(A)) = 0$ $\iff$ $\text{rank}(A) = n$).

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Plan the week after the Easter break

Wednesday we shall introduce and study

- *eigenvectors*, *eigenvalues* and *eigenspaces* of square matrices,
- the *characteristic polynomial* of a square matrix.

Sections 5.1–5.2 in “Linear Algebras and Its Applications” (pages 265—281).

Thursday we shall look at *diagonalizable* matrices. Section 5.3 in “Linear Algebras and Its Applications” (pages 281—288).

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