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TMA4115 - Calculus 3

Lecture 2, Jan 17

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Course web page

Information about the course can be found at `http://wiki.math.ntnu.no/tma4115/2013v`.

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Review of yesterday's lecture

Yesterday we introduced *complex numbers* and studied

- the *real part*, the *imaginary part*, the *absolute value* (or *modulus*), and the *argument* of a complex number,
- addition and multiplication of complex numbers,
- and *complex conjugation*.

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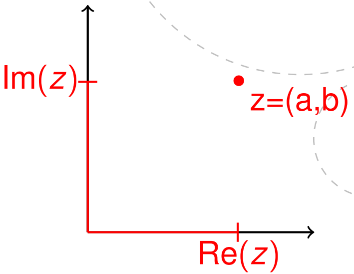
Today's lecture

Today we will study

- *polar representation* of complex numbers,
- *de Moivre's Theorem*,
- how complex numbers can be used to derive trigonometric identities,
- roots of complex numbers.

Complex numbers

- A complex number is a number which can be written as $a + ib$ where a and b are real numbers and i satisfies $i^2 = -1$.
- A complex number $z = a + ib$ can be represented as the point (a, b) in the plane.
- If $z = a + ib$, then a is called the *real part* of z and is denoted by $\mathrm{Re}(z)$.
- b is called the *imaginary part* of z and is denoted by $\mathrm{Im}(z)$.



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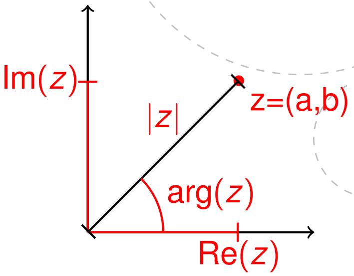
Complex numbers

- If $z = a + ib$, then the length $\sqrt{a^2 + b^2}$ of the line from $(0, 0)$ to (a, b) is called the *modulus* or the *absolute value* of z and is denoted by $|z|$.
- The angle between the line through $(0, 0)$ and (a, b) and the positive part of the real axis is called the *argument* of z and is denoted by $\arg(z)$.



Complex numbers

- $\arg(z)$ is not unique. If $\theta = \arg(z)$, then also $\theta + 2\pi = \arg(z)$. If we want to be precise, then $\arg(z)$ is really the set of all angles θ which satisfies that if we rotate the positive part of the real axis by θ, then it lands on the line through $(0, 0)$ and (a, b).
- The unique value of $\arg(z)$ in the interval $(-\pi, \pi]$ is called the *principal argument* of z and is denoted by $\mathrm{Arg}(z)$.
- Notice that $\arg(z)$ and $\mathrm{Arg}(z)$ are not defined if $z = (0, 0)$.

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Addition of complex numbers

- If $z_1 = a + bi$ and $z_2 = c + di$, then $z_1 + z_2 = (a + c) + (b + d)i$.
- $\text{Re}(z_1 + z_2) = \text{Re}(z_1) + \text{Re}(z_2)$ and $\text{Im}(z_1 + z_2) = \text{Im}(z_1) + \text{Im}(z_2)$.
- $z_1 + z_2 = z_2 + z_1$.
- $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$.
- $|z_1 + z_2| \leq |z_1| + |z_2|$.

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Multiplication of complex numbers

- If $z_1 = a + bi$ and $z_2 = c + di$, then $z_1 z_2 = (a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i$.
- $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$ and $|z_1 z_2| = |z_1||z_2|$.
- $z_1 z_2 = z_2 z_1$.
- $(z_1 z_2) z_3 = z_1 (z_2 z_3)$.
- $z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$.

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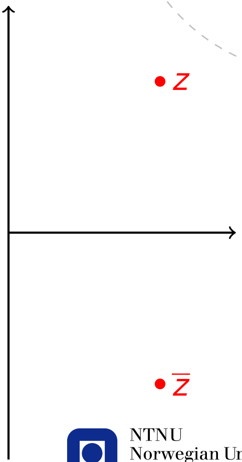
Complex conjugation

- When $z = a + bi$ is a complex number, then the number $a - bi$ is called the *conjugate* of z and is denoted by $\overline{z}$.
- $\mathrm{Re}(\overline{z}) = \mathrm{Re}(z)$ and $\mathrm{Im}(\overline{z}) = -\mathrm{Im}(z)$.
- We get $\overline{z}$ by reflecting z in the real line.
- $|\overline{z}| = |z|$ and $\arg(\overline{z}) = -\arg(z)$.



Complex conjugation

- $z = \bar{z}$ if and only if $\mathrm{Im}(z) = 0$.
- $z = -\bar{z}$ if and only if $\mathrm{Re}(z) = 0$.
- $\overline{z + w} = \bar{z} + \bar{w}$.
- $\overline{zw} = \bar{z}\,\bar{w}$.
- $z\bar{z} = (a+bi)(a-bi) = a^2 - abi + abi - b^2 i^2 = a^2 + b^2 = |z|^2$.

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Division of complex numbers

- If z and w are complex numbers and $w \neq 0$, then $\dfrac{z}{w} = \dfrac{z\overline{w}}{w\overline{w}} = \dfrac{z\overline{w}}{|w|^2}$.
- $\dfrac{a+bi}{c+di} = \dfrac{(a+bi)(c-di)}{c^2+d^2} = \dfrac{ac+bd}{c^2+d^2} + \dfrac{bc-ad}{c^2+d^2}i$.
- $\arg\left(\dfrac{z}{w}\right) = \arg(z) - \arg(w)$ and $\left|\dfrac{z}{w}\right| = \dfrac{|z|}{|w|}$.

Example

$$\frac{1+i}{2-i} = \frac{(1+i)(2+i)}{(2-i)(2+i)} = \frac{2-1+3i}{5} = \frac{1}{5} + \frac{3}{5}i.$$

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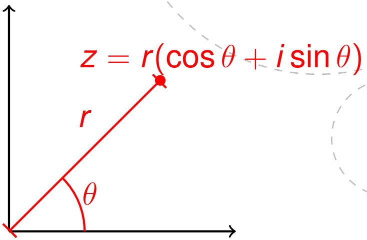
Polar representation

- Every complex number z can be written on the form $r(\cos\theta + i\sin\theta)$ where r and θ are real numbers and $r \geq 0$. This is called the *polar form* of z.
- Notice that if $z = r(\cos\theta + i\sin\theta)$, then $|z| = r$ and $\arg(z) = \theta$.



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Multiplication and division using polar representations

Recall that

- $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$ and $|z_1 z_2| = |z_1||z_2|$, and that
- $\arg\left(\frac{z}{w}\right) = \arg(z) - \arg(w)$ and $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$.

It follows that if $z = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then

- $z_1 z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$ and
- $\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))$ (provided $r_2 \neq 0$).

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Exercise 46, page xviii

Express each of the complex numbers $z = 3 + i\sqrt{3}$ and $w = -1 + i\sqrt{3}$ in polar form. Use these expressions to calculate zw and z/w.

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Solution

$$|z| = \sqrt{3^2 + (\sqrt{3})^2} = \sqrt{12},$$

$$\text{Arg}(z) = \arctan\left(\frac{\text{Im}(z)}{\text{Re}(z)}\right) = \arctan\left(\frac{\sqrt{3}}{3}\right) = \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6},$$

$$|w| = \sqrt{(-1^2) + (\sqrt{3})^2} = \sqrt{4} = 2, \text{ and}$$

$$\text{Arg}(w) = \arccos\left(\frac{\text{Re}(w)}{|w|}\right) = \arccos\left(\frac{-1}{2}\right) = \frac{2\pi}{3},$$

so $|zw| = |z||w| = (\sqrt{12})2 = 4\sqrt{3}$,
$\arg(zw) = \text{Arg}(z) + \text{Arg}(w) = \frac{\pi}{6} + \frac{2\pi}{3} = \frac{5\pi}{6}$,
$|z/w| = |z|/|w| = \sqrt{12}/2 = \sqrt{3}$, and
$\arg(z/w) = \text{Arg}(z) - \text{Arg}(w) = \frac{\pi}{6} - \frac{2\pi}{3} = \frac{-\pi}{2}$, from which it follows that $zw = 4\sqrt{3}(\cos(5\pi/6) + i\sin(5\pi/6)) =$
$4\sqrt{3}\left(\frac{-\sqrt{3}}{2} + \frac{1}{2}i\right) = -6 + 2\sqrt{(3)}i$, and $z/w =$
$\sqrt{3}(\cos(-\pi/2) + \sin(-\pi/2)) = -\sqrt{3}i$.

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Complex numbers and trigonometric identities

Polar representations of complex numbers can be used to derive trigonometric identities like

- $\cos(\theta_1 + \theta_2) = \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)$
- $\sin(\theta_1 + \theta_2) = \cos(\theta_1)\sin(\theta_2) + \sin(\theta_1)\cos(\theta_2)$

Proof

We have that

$$
\begin{aligned}
&\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)\\
&\quad=\big(\cos\theta_1+i\sin\theta_1\big)\big(\cos\theta_2+i\sin\theta_2\big)\\
&\quad=\cos\theta_1\cos\theta_2+i\cos\theta_1\sin\theta_2+i\sin\theta_1\cos\theta_2-\sin\theta_1\sin\theta_2\\
&\quad=\cos\theta_1\cos\theta_2-\sin\theta_1\sin\theta_2+i(\cos\theta_1\sin\theta_2+\sin\theta_1\cos\theta_2)
\end{aligned}
$$

from which it follows that
$\cos(\theta_1+\theta_2)=\cos(\theta_1)\cos(\theta_2)-\sin(\theta_1)\sin(\theta_2)$ and
$\sin(\theta_1+\theta_2)=\cos(\theta_1)\sin(\theta_2)+\sin(\theta_1)\cos(\theta_2)$.

de Moivre’s Theorem

- Recall that $\arg(zw) = \arg(z) + \arg(w)$ and $|zw| = |z||w|$.
- It follows that $(r(\cos\theta + i\sin\theta))^n = r^n(\cos(n\theta) + i\sin(n\theta))$.
- We have in particular that

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta).$$

- This formula is know as *de Moivre’s formula*.

Complex numbers and trigonometric identities

Polar representations of complex numbers can be used to derive trigonometric identities like

- $\cos(\theta_1 + \theta_2) = \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)$
- $\sin(\theta_1 + \theta_2) = \cos(\theta_1)\sin(\theta_2) + \sin(\theta_1)\cos(\theta_2)$
- $\cos(3\theta) = \cos^3(\theta) - 3\cos(\theta)\sin^2(\theta)$
- $\sin(3\theta) = 3\cos^2(\theta)\sin(\theta) - \sin^3(\theta)$

Proof

It follows from de Moivre's formula and the binomial formula that

$$\begin{aligned}
&\cos(3\theta) + i\sin(3\theta) \\
&\quad = \cos^3(\theta) + 3i\cos^2(\theta)\sin(\theta) + 3i^2\cos(\theta)\sin^2(\theta) + i^3\sin^3(\theta) \\
&\quad = \cos^3(\theta) - 3\cos(\theta)\sin^2(\theta) + i\big(3\cos^2(\theta)\sin(\theta) - \sin^3(\theta)\big)
\end{aligned}$$

from which it follows that $\cos(3\theta) = \cos^3(\theta) - 3\cos(\theta)\sin^2(\theta)$ and $\sin(3\theta) = 3\cos^2(\theta)\sin(\theta) - \sin^3(\theta)$.

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Roots of complex numbers

- If z is a complex number and n is a positive integer, then an *n'th root* of z is a complex number w satisfying $w^n = z$.
- Every complex number different from 0 has n different n'th roots.

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Example

Let us find the 3 cube roots (3rd roots) of $z = -2 - 2i$.
$|z| = \sqrt{(-2)^2 + (-2)^2} = \sqrt{8}$ and $\text{Arg}(z) = \frac{-3\pi}{4}$, so it follows from de Movire's formula that

$$w^3 = z \iff |w|^3 = |z| = \sqrt{8} \text{ and}$$

$$3\arg(w) = \arg(z) = \frac{-3\pi}{4} + 2\pi k, \ k \in \mathbb{Z}$$

$$\iff |w| = \sqrt[3]{\sqrt{8}} = 8^{1/6} = \sqrt{2} \text{ and}$$

$$\arg w = \frac{-\pi}{4} + \frac{2\pi k}{3}, \ k \in \mathbb{Z}.$$

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Example

So the 3 cube roots of $z = -2 - 2i$ are

$$w_0 = \sqrt{2}\big(\cos(-\pi/4) + i\sin(-\pi/4)\big)$$
$$= \sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = 1 - i$$
$$w_1 = \sqrt{2}\big(\cos(5\pi/12) + i\sin(5\pi/12)\big)$$
$$= \sqrt{2}\left(\frac{\sqrt{3}-1}{2\sqrt{2}} + i\frac{1+\sqrt{3}}{2\sqrt{2}}i\right) = \frac{\sqrt{3}-1}{2} + i\frac{1+\sqrt{3}}{2}$$
$$w_2 = \sqrt{2}\big(\cos(13\pi/12) + i\sin(13\pi/12)\big)$$
$$= \sqrt{2}\left(-\frac{1+\sqrt{3}}{2\sqrt{2}} + i\frac{\sqrt{3}-1}{2\sqrt{2}}i\right) = -\frac{1+\sqrt{3}}{2} + i\frac{\sqrt{3}-1}{2}$$

Roots of complex numbers

If z is a complex number different from 0 and n is a positive integer, then the n'th roots of z are

$$w_k = |z|^{1/n} \left(\cos\left(\frac{\arg(z) + 2\pi k}{n} \right) + i \sin\left(\frac{\arg(z) + 2\pi k}{n} \right) \right)$$

where $k = 0, 1, \ldots, n - 1$.

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Roots of unity

- If n is a positive integer, then a n'th root of 1 is called a *n'th root of unity*.
- The n n'th roots of unity are

$$u_n = \cos\left(\frac{2\pi k}{n}\right) + i\sin\left(\frac{2\pi k}{n}\right)$$

for $k = 0, 1, \ldots, n-1$.

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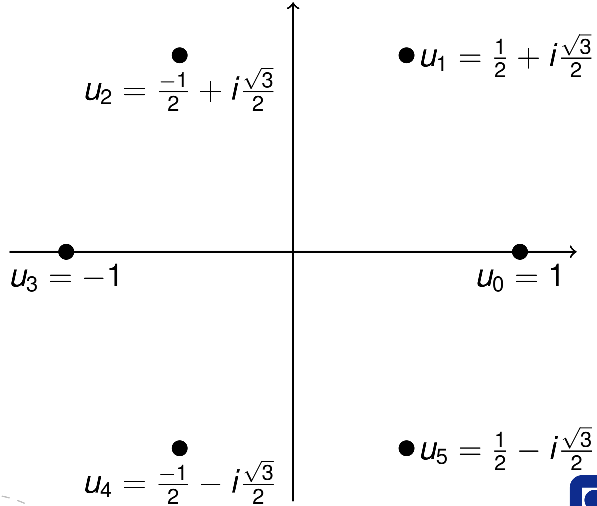
The 6 6'th root of unity



Roots of complex numbers

If z is a complex number different from 0, n is a positive integer and w is an n'th root of z, then the other $n-1$ n'th roots of z are $wu_1, wu_2, \dots wu_{n-1}$ where $u_1, u_2, \dots u_{n-1}$ are the $n-1$ n'th roots of unity different from 1.

Problem 1 from the exam from August 2012

Write all of the solutions of $z^3 = 1$ in the form $z = x + iy$.
Write the solutions of $z^3 = \frac{-3+i}{\sqrt{2}(2+i)}$ in the form $z = x + iy$ and draw the solutions in the complex plane.

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Solution

The solutions of $z^3 = 1$ are the 3 cube roots of unity

$$u_0 = \cos(0) + i\sin(0) = 1$$

$$u_1 = \cos(2\pi/3) + i\sin(2\pi/3) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$u_2 = \cos(4\pi/3) + i\sin(4\pi/3) = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$

$$\frac{-3+i}{\sqrt{2}(2+i)} = \frac{(-3+i)(2-i)}{\sqrt{2}(2+i)(2-i)} = \frac{-6+3i+2i-1}{(\sqrt{2})5}$$

$$= \frac{-5+5i}{\sqrt{25}} = \frac{-1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$$

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Solution

So $\left|\frac{-3+i}{\sqrt{2}(2+i)}\right| = \left|\frac{-1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right| = \sqrt{(-1/\sqrt{2})^2 + (1/\sqrt{2})^2} = 1$ and $\operatorname{Arg}\left(\frac{-3+i}{\sqrt{2}(2+i)}\right) = \operatorname{Arg}\left(\frac{-1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \frac{3\pi}{4}$. It follows that the solutions of $z^3 = \frac{-3+i}{\sqrt{2}(2+i)}$ are

$$z_0 = \cos(\pi/4) + i\sin(\pi/4) = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$$

$$z_1 = z_0 u_1 = \left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$

$$= \frac{-1-\sqrt{3}}{2\sqrt{2}} + i\frac{\sqrt{3}-1}{2\sqrt{2}}$$

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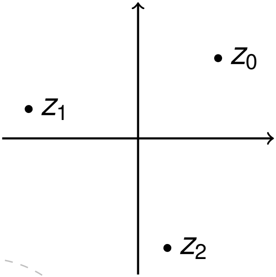
Solution

and
$$z_2 = z_0 u_2 = \left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$$
$$= \frac{-1+\sqrt{3}}{2\sqrt{2}} + i\frac{-\sqrt{3}-1}{2\sqrt{2}}.$$



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Problem 1 from the exam from June 2009

Find all complex numbers $z = x + iy$ which satisfy the equality $|z + 1 - i\sqrt{3}| = |z - 1 + i\sqrt{3}|$. Draw the solutions in a diagram.

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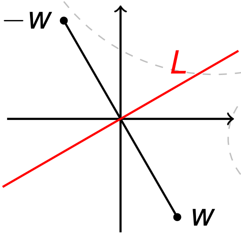
Solution

Let $w = 1 - i\sqrt{3}$. Then
$|z + 1 - i\sqrt{3}| = |z - (-w)|$ is the distance between z and $-w$, and
$|z - w| = |z - 1 + i\sqrt{3}|$ is the distance between z and w.
So $|z + 1 - i\sqrt{3}| = |z - 1 + i\sqrt{3}|$ if and only if z has the same to $-w$ as it has to w.
The set of complex numbers z which satisfy the equality
$|z + 1 - i\sqrt{3}| = |z - 1 + i\sqrt{3}|$ is therefore the set of points that lie on the line L which goes through 0 and which is perpendicular to the line between w and $-w$.



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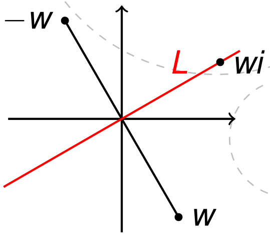
Solution

If we rotate the point w by 90 degrees, then we get a point that lies on the line L. Since rotating w by 90 degrees is the same as multiplying w by i, it follows that $wi = \sqrt{3} + i$ lies on the line L.
So the points that lie on the line L are the points of the form $t(\sqrt{3} + i)$ where t is a real number.
Thus, $|z + 1 - i\sqrt{3}| = |z - 1 + i\sqrt{3}|$ if and only if $z = t(\sqrt{3} + i)$ for some real number t.



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Next week's lectures

Wednesday we shall

- use complex numbers to solve polynomial equations,
- look at *the fundamental theorem of algebra*,
- introduce *the complex exponential function*,
- and study extensions of trigonometric functions to the complex numbers.

Thursday we shall

- study second-order differential equations,
- introduce the *Wronskian*,
- completely solve second-order homogeneous linear differential equations with constant coefficients.