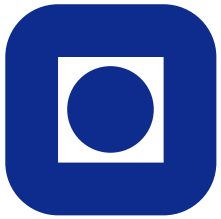

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TMA4115 - Calculus 3

Lecture 19, March 20

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Review of last week's lecture

Last week we looked at

- abstract *vector spaces* and *subspaces*,
- *null spaces* and *column spaces* of matrices,
- *linear transformations* between abstract vector spaces,
- *kernels* and *ranges* of linear transformations,
- *linear independence* and *linear dependence* in abstract vector spaces.
- *bases* of vector spaces,
- *coordinate systems* in vector spaces relative to bases.

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Today's lecture

Today we shall introduce and study

- the *dimension* of a vector space,
- the *rank* of a matrix.

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Bases of vector spaces

Recall that an indexed set $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ of vectors in a vector space V is a *basis* for V if

1. $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ is linearly independent,
2. $V = \text{Span}\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$.

Recall also that if $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_k \in \mathbb{R}^n$ then

- $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_k\}$ is linearly independent if and only if the matrix $[\mathbf{b}_1\ \mathbf{b}_2\ \ldots\ \mathbf{b}_k]$ has a pivot position in every column,
- $\text{Span}\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_k\} = \mathbb{R}^n$ if and only if the matrix $[\mathbf{b}_1\ \mathbf{b}_2\ \ldots\ \mathbf{b}_k]$ has a pivot position in every row.

It follows that every basis of $\mathbb{R}^n$ consists of exactly n vectors.

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The coordinate mapping

Recall also that the following theorem holds for every vector space.

Theorem 8

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Then the coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is a one-to-one and onto linear transformation from V to $\mathbb{R}^n$.

It is therefore natural to expect that if a vector space has a basic of n vectors, then every basis of V must consist of exactly n vectors.

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The dimension of a vector space

Theorem 9

If a vector space V has a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, then any set in V containing more than n vectors must be linearly dependent.

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Proof of Theorem 9

Let $p > n$ and let $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ be a set in V with p vectors. The matrix $\begin{bmatrix}[\mathbf{u}_1]_{\mathcal{B}} & [\mathbf{u}_2]_{\mathcal{B}} & \ldots & [\mathbf{u}_p]_{\mathcal{B}}\end{bmatrix}$ has more columns than rows, and thus cannot have a pivot position in every row. It follows that the set $\{[\mathbf{u}_1]_{\mathcal{B}}, [\mathbf{u}_2]_{\mathcal{B}}, \ldots, [\mathbf{u}_p]_{\mathcal{B}}\}$ is linearly dependent. Thus there exist scalars $c_1, c_2, \ldots, c_p$, not all equal to zero, such that

$$\mathbf{0} = c_1[\mathbf{u}_1]_{\mathcal{B}} + c_2[\mathbf{u}_2]_{\mathcal{B}} + \cdots + c_p[\mathbf{u}_p]_{\mathcal{B}} = [c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_p\mathbf{u}_p]_{\mathcal{B}}.$$

Thus $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_p\mathbf{u}_p = 0\mathbf{b}_1 + 0\mathbf{b}_2 + \cdots + 0\mathbf{b}_n = \mathbf{0}$. This shows that the set $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is linearly dependent.

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The dimension of a vector space

Theorem 10

If a vector space has a basis of n vectors, then every basis of V must consist of exactly n vectors.

Proof of Theorem 10

Suppose that $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ and $\{\mathbf{b}_1, \ldots, \mathbf{b}_m\}$ are both bases for V. Then $n \leq m$ because $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ is linearly independent and $\{\mathbf{b}_1, \ldots, \mathbf{b}_m\}$ is a basis for V, and $m \leq n$ because $\{\mathbf{b}_1, \ldots, \mathbf{b}_m\}$ is linearly independent and $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ is a basis for V.
Thus $m = n$.

The dimension of a vector space

- If V is spanned by a finite set, then V is said to be *finite-dimensional*.
- If V is a finite dimensional vector space, then the *dimension* of V, written as $\dim(V)$, is the number of vectors in a basis for V.
- The dimension of the zero vector space $\{\mathbf{0}\}$ is defined to be zero.
- If V is not spanned by a finite set, then V is said to be *infinite-dimensional*.

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The dimension of some vector spaces

- Since $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is a basis for $\mathbb{R}^n$, the vector space $\mathbb{R}^n$ is finite-dimensional and $\dim(\mathbb{R}^n) = n$.
- Since $\{1, t, \ldots, t^n\}$ is a basis for $\mathbb{P}_n$, the vector space $\mathbb{P}_n$ is finite-dimensional and $\dim(\mathbb{P}_n) = n + 1$.
- The vector space $\mathbb{P}$ is infinite-dimensional, because if $\{\mathbf{p}_1, \ldots, \mathbf{p}_k\}$ is a set of elements of $\mathbb{P}$, then there is an n such that $\{\mathbf{p}_1, \ldots, \mathbf{p}_k\} \subseteq \mathbb{P}_n$, and then $\text{Span}\{\mathbf{p}_1, \ldots, \mathbf{p}_k\} \subseteq \mathbb{P}_n$, from which it follows that $\{\mathbf{p}_1, \ldots, \mathbf{p}_k\}$ does not generate $\mathbb{P}$ (because $\mathbb{P}_n \neq \mathbb{P}$).

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Every vector space of dimension n is isomorphic to $\mathbb{R}^n$

The theorem

Theorem 8

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Then the coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is a one-to-one and onto linear transformation from V to $\mathbb{R}^n$.

says that if V is a vector space of dimension n, then V is *isomorphic* to $\mathbb{R}^n$.

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Subspaces of a finite-dimensional space

Theorem 11

Let H be a subspace of a finite-dimensional vector space V.
Then H is finite-dimensional and $\dim(H) \leq \dim(V)$.
In addition, if $H \neq \{\mathbf{0}\}$, then any linearly independent set in H can be expanded to a basic for H.

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Proof of Theorem 11

Let $n = \dim(V)$. If $H = \{\mathbf{0}\}$, then H is finite-dimensional and $\dim(H) = 0 \leq n = \dim(V)$.
Assume that $H \neq \{\mathbf{0}\}$ and let $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ be a linear independent subset of H. If $\text{Span}\{\mathbf{u}_1, \ldots, \mathbf{u}_k\} = H$, then $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is a basis for H. Otherwise there is a vector $\mathbf{u}_{k+1}$ in H which is not in $\text{Span}\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$, and then $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}\}$ is linearly independent.
We can continue adding vectors like this as long as $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_m\}$ is linearly independent. Since no set in V containing more than n vectors can be linearly independent, if follows that $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ most span H for some $m \leq n$. It follows that H is finite-dimensional and $\dim(H) \leq \dim(V)$.

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Example

Let us find the dimension of the subspace

$$H = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : a, b, c, d \text{ in } \mathbb{R} \text{ and } a + b + c = 0 \text{ and } 2a - d = 0 \right\}$$

of $\mathbb{R}^4$.

Solution

$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}$ is in H, so $H \neq \{\mathbf{0}\}$.

$\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}$ is in H, but not in $\text{Span}\{\mathbf{v}_1\}$. So $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent.

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Solution (cont.)

If $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ is in H, then

$$-b\mathbf{v}_1 - c\mathbf{v}_2 = -b\begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix} - c\begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -b-c \\ b \\ c \\ -2b-2c \end{bmatrix}$$

$$= \begin{bmatrix} a \\ b \\ c \\ 2a \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}.$$

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Solution (cont.)

So Span$\{\mathbf{v}_1, \mathbf{v}_2\} = H$. It follows that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for H, and thus that $\dim(H) = 2$.

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Example

Let us find the dimension of the subspace

$$H = \left\{ \begin{bmatrix} 2r + s + 3t \\ -r + 4s + 3t \\ 0 \end{bmatrix} : r, s, t \text{ in } \mathbb{R} \right\}$$

of $\mathbb{R}^3$.

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Solution

Let $\mathbf{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}$.

If r, s, t are in $\mathbb{R}^3$, then

$$\begin{bmatrix} 2r + s + 3t \\ -r + 4s + 3t \\ 0 \end{bmatrix} = r \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} = r\mathbf{v}_1 + s\mathbf{v}_2 + t\mathbf{v}_3$$

This shows that $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = H$.

$\mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2$, so $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\} = H$.

$\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent and thus a basis for H, so $\dim(H) = 2$.

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The basis theorem

Theorem 12

Let V be a p-dimensional vector space, where $p \geq 1$.

1. Any linearly independent set of exactly p elements in V is automatically a basis for V
2. Any set of exactly p elements that spans V is automatically a basis for V.

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Proof of Theorem 12

1. Let S be a linearly independent set consisting of p vectors in V. Then S can be extended to a basis for V. This basis must contain exactly p vectors, so it is equal to the original set S which is therefore a basis for V.
2. Let S be a of p vectors that spans V. Then some subset of S is a basis for V. This basis must contain exactly p vectors, so it is equal to the original set S which is therefore a basis for V.

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Problem 4 from the exam from December 2008

Assume that the vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly independent. Show that the vectors $\mathbf{w}_1 = \mathbf{v}_1 + \mathbf{v}_2$, $\mathbf{w}_2 = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$, $\mathbf{w}_3 = \mathbf{v}_1 - \mathbf{v}_2$ are linearly independent.

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Solution

Let $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for H, so $\dim(H) = 3$.

$$\mathbf{v}_1 = \frac{1}{2}(\mathbf{w}_1 + \mathbf{w}_3), \quad \mathbf{v}_2 = \frac{1}{2}(\mathbf{w}_1 - \mathbf{w}_3), \quad \mathbf{v}_3 = \mathbf{w}_2 - \mathbf{w}_1,$$

so $\text{Span}\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\} = H$. Since $\dim(H) = 3$, it follows that $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is a basis for H and therefore linearly independent.

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The dimension of the column space of a matrix

Recall that if $A = [\mathbf{a}_1\ \mathbf{a}_2 \ldots \mathbf{a}_n]$ is an $m \times n$ matrix, then $\text{Col}(A) = \text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$.
Recall also that the pivot columns of A form a basis for $\text{Col}(A)$.
It follows that the dimension of $\text{Col}(A)$ is the number of pivot positions in A.

The dimension of the null space of a matrix

Recall that if A is an $m \times n$ matrix, then
$\text{Nul}(A) = \{\mathbf{x} : \mathbf{x} \text{ is in } \mathbb{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\}$.
Recall also that we can find a basis for $\text{Nul}(A)$ by

1. row reducing A to its reduced echelon form,
2. writing the solutions to the equation $A\mathbf{x} = \mathbf{0}$ as a linear combinations of vectors using the free variables as parameters,
3. forming a basis for $\text{Nul}(A)$ by taken the set of the vectors we use to write the solutions to the equation $A\mathbf{x} = \mathbf{0}$.

It follows that the dimension of $\text{Nul}(A)$ is the number of nonpivot columns of A.

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The row space of a matrix

- Let A be an $m \times n$ matrix.
- Each row of A can be identified with a vector in $\mathbb{R}^n$.
- The set of all linear combination of the row vectors is called the *row space* of A and is denoted by $\text{Row}(A)$.
- $\text{Row}(A) = \text{Col}(A^T)$.

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Finding a basis for the row space of a matrix

Theorem 13

Let A and B be matrices.

1. If A and B are row equivalent, then $\text{Row}(A) = \text{Row}(B)$.
2. If B is in echelon form, then the nonzero rows of B form a basis for $\text{Row}(B)$.

It follows that the dimension of $\text{Row}(A)$ is the number of pivot positions in A.

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Example

Let $A = \begin{bmatrix} 1 & 4 & -1 \\ -1 & -4 & 2 \\ 2 & 8 & -2 \end{bmatrix}$.

The the reduced echelon form of A is $\begin{bmatrix} 1 & 4 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. It follows that the pivot columns of A are the first and the third columns which therefore form a basis of $\mathrm{Col}(A)$. Thus, $\dim(\mathrm{Col}(A)) = 2$.

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Example (cont.)

It also follows that the general solution of the equation $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = t \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix}$ where t is a free parameter, and thus that $\left\{ \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis of $\text{Nul}(A)$. So $\dim(\text{Nul}(A)) = 1$.

Finally, we see that the nonzero rows of the reduced echelon form of A are $(1, 4, 0)$ and $(0, 0, 1)$, so $\{(1, 4, 0), (0, 0, 1)\}$ is a basis for $\text{Row}(A)$. Thus $\dim(\text{Row}(A)) = 2$.

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The rank of a matrix

Let A be a matrix. Then the *rank* of A, written rank(A), is the dimension of the column space Col(A) of A.

Theorem 14

Let A be an $m \times n$ matrix. Then the following numbers are equal:

- rank(A)
- dim(Col(A))
- dim(Row(A))
- $n -$ dim(Nul(A))
- The number of pivot positions in A.

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Example

If A is 3×4 matrix with a 3-dimensional null space, what is rank(A)?
Solution: $\text{rank}(A) = n - \dim(\text{Nul}(A)) = 4 - 3 = 1$.

Example

Suppose we have a homogeneous linear system consisting of 25 equations in 28 variables. Can we be certain that there the system has 3 linearly independent solutions?
Let A be the coefficient matrix of the system. $\text{rank}(A) \leq 25$, so $\dim(\text{Nul}(A)) = 28 - \text{rank}(A) \geq 28 - 25 = 3$. It follows that the system has (at least) 3 linearly independent solutions.

Tomorrow's lecture

Tomorrow we shall look at *Markov chains*.
Section 4.9 in "Linear Algebras and Its Applications" (pages 253–264).

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