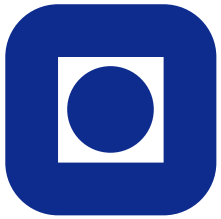

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TMA4115 - Calculus 3

Lecture 18, March 14

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Yesterday's lecture

Yesterday we introduced and studied

- abstract *vector spaces* and *subspaces*,
- *null spaces* and *column spaces* of matrices,
- *linear transformations* between abstract vector spaces,
- *kernels* and *ranges* of linear transformations,
- *linear independence* and *linear dependence* in abstract vector spaces.

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Today's lecture

Today we shall introduce and study

- *bases* of vector spaces,
- *coordinate systems* in vector spaces relative to bases.

Vector spaces

Definition

A *vector space* is a nonempty set V of objects, called vectors, on which are defined two operations, called addition and multiplication by scalars (real numbers), subject to the ten axioms listed on the next slide.

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1. The *sum* of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u}+\mathbf{v}$, is in V.
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.
3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.
4. There is a *zero* vector $\mathbf{0}$ in V such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$.
5. For each $\mathbf{u}$ in V, there is a vector $-\mathbf{u}$ in V such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.
6. The *scalar multiple* of $\mathbf{u}$ by c, denoted by $c\mathbf{u}$, is in V.
7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$.
8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$.
9. $c(d\mathbf{u})=(cd)\mathbf{u}$.
10. $1\mathbf{u}=\mathbf{u}$.

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Subspaces

Definition

A *subspace* of a vector space V is a subset H of V that has 3 properties:

1. The zero vector of V is in H.
2. If $\mathbf{u}$ and $\mathbf{v}$ are in H, then $\mathbf{u} + \mathbf{v}$ is in H.
3. If $\mathbf{u}$ is in H and c is a scalar, then $c\mathbf{u}$ is in H.

Every subspace is a vector space. Conversely, every vector space is a subspace (of itself and possibly of other larger spaces).

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Linear independence

Let V be a vector space, and let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be an indexed set of vectors in V.

- $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is said to be *linearly independent* if the vector equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$ has only the trivial solution.
- $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is said to be *linearly dependent* if there exist scalars $c_1, c_2, \ldots, c_n$ not all equal to zero such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$.

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Linear independence in $\mathbb{R}^n$

- Recall that if $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ are vectors in $\mathbb{R}^n$, then the indexed set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is linearly independent if and only if the matrix $[\mathbf{v}_1\ \mathbf{v}_2 \ldots \mathbf{v}_p]$ has a pivot position in every column.
- There is no similar method to determine if a indexed set of vectors in an arbitrary vector space is linearly independent.

Linear dependence

Theorem 4

An indexed set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ of two or more vectors in a vector set V is linearly dependent if and only if $\mathbf{v}_1 = \mathbf{0}$ or some $\mathbf{v}_j$, with $j > 1$, is a linear combination of the preceding vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{j-1}$.

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Bases

Definition

Let H be a subspace of a vector space V. An indexed set $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ of vectors in H is a *basis* for H if

1. $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ is linearly independent,
2. $H = \text{Span}\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$.

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The standard basis for $\mathbb{R}^n$

- Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ be the columns of I_n.
- That is $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$, $\ldots$, $\mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$.
- Since I_n is invertible, it follows from the inverse matrix theorem that $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ spans $\mathbb{R}^n$ and is linearly independent.
- So $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is a basis for $\mathbb{R}^n$.
- $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is called the *standard basis* for $\mathbb{R}^n$.

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Bases of $\mathbb{R}^n$

- Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be vectors in $\mathbb{R}^n$.
- If the matrix $[\mathbf{v}_1\ \mathbf{v}_2 \ldots \mathbf{v}_n]$ is invertible, then it follows from the inverse matrix theorem that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ spans $\mathbb{R}^n$ and is linearly independent.
- So $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for $\mathbb{R}^n$ if the matrix $[\mathbf{v}_1\ \mathbf{v}_2 \ldots \mathbf{v}_n]$ is invertible.

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The standard basis for $\mathbb{P}_n$

- Recall that $\mathbb{P}_n$ denotes the vector space of polynomials with real coefficients of degree at most n.
- Consider the set of polynomials $\{1, t, t^2, \ldots, t^n\}$ in $\mathbb{P}_n$.
- Since every element of $\mathbb{P}_n$ has the form $a_n t^n + a_{n_1} t^{n-1} + \cdots + a_1 t + a_0$, it follows that $\text{Span}\{1, t, t^2, \ldots, t^n\} = \mathbb{P}_n$.
- If $c_0 \cdot 1 + c_1 t + \cdots + c_n t^n = 0$, then the coefficients $c_0, c_1, \ldots, c_n$ must all be equal to zero, so $\{1, t, t^2, \ldots, t^n\}$ is linearly independent, and thus a basis for $\mathbb{P}_n$.
- $\{1, t, t^2, \ldots, t^n\}$ is called the *standard basis* for $\mathbb{P}_n$.

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Bases for the solution set of homogeneous linear second-order differential equations

- Let $y'' + p(t)y' + q(t)y = 0$ be a homogenous second-order linear differential equation.
- Recall that the set of solutions to $y'' + p(t)y' + q(t)y = 0$ is the kernel of the linear transformation that maps a real-valued twice continuously differentiable function f to $f'' + p(t)f' + q(t)f$.
- Recall that if y_1 and y_2 are two linearly independent solutions to $y'' + p(t)y' + q(t)y = 0$, then $\{y_1, y_2\}$ is called a fundamental set of solutions.

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Bases for the solution set of homogeneous linear second-order differential equations

- Recall also that if $\{y_1, y_2\}$ is a fundamental set of solutions, then every solution of $y'' + p(t)y' + q(t)y = 0$ is a linear combination of y_1 and y_2.
- So a fundamental set of solutions is a basis for the solutions set of $y'' + p(t)y' + q(t)y = 0$.

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Example

Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix}$, and let $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

Let us find a basis for H.

Notice that $\mathbf{v}_1 + 5\mathbf{v}_2 = \mathbf{v}_3$. It follows that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent, and thus not a basis for H.

If $\mathbf{v}$ is in H, then there are scalars c_1, c_2, c_3 such that

$$\begin{aligned} \mathbf{v} &= c_1\mathbf{v}_1 + c_2\mathbf{v}_2 2 + c_3\mathbf{v}_3 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 2 + c_3(\mathbf{v}_1 + 5\mathbf{v}_2) \\ &= (c_1 + c_3)\mathbf{v}_1 + (c_2 + 5c_3)\mathbf{v}_2. \end{aligned}$$

So $\mathbf{v}$ is in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. This shows that $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\} = H$.

We furthermore have that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent. So $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for H.

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The spanning set theorem

Theorem 5

Let V be a vector space, let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be an indexed set of vectors in V, and let $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$.

1. If one of the vectors in $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, say $\mathbf{v}_k$, is a linear combination of the remaining vectors in $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, then the set formed from $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ by removing $\mathbf{v}_k$ still spans H.
2. If $H \neq \{\mathbf{0}\}$, then some subset of $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for H.

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Proof of Theorem 5

1. Suppose that $\mathbf{v}_k = a_1\mathbf{v}_1 + \cdots + a_{k-1}\mathbf{v}_{k-1} + a_{k+1}\mathbf{v}_{k+1} + \cdots + a_n\mathbf{v}_n$. If $\mathbf{v}$ is in H, then there are scalars $c_1, c_2, \ldots, c_n$ such that

$$\begin{aligned}
\mathbf{v} &= c_1\mathbf{v}_1 + \cdots + c_{k-1}\mathbf{v}_{k-1} + c_k\mathbf{v}_k + c_{k+1}\mathbf{v}_{k+1} + \cdots + c_n\mathbf{v}_n \\
&= c_1\mathbf{v}_1 + \cdots + c_{k-1}\mathbf{v}_{k-1} + c_k(a_1\mathbf{v}_1 + \cdots + a_{k-1}\mathbf{v}_{k-1} \\
&\quad + a_{k+1}\mathbf{v}_{k+1} + \cdots + a_n\mathbf{v}_n) + c_{k+1}\mathbf{v}_{k+1} + \cdots + c_n\mathbf{v}_n \\
&= (c_1 + c_k a_1)\mathbf{v}_1 + \cdots + (c_{k-1} + c_k a_{k-1})\mathbf{v}_{k-1} \\
&\quad + (c_{k+1} + c_k a_{k+1})\mathbf{v}_{k+1} + \cdots + (c_n + c_k a_n)\mathbf{v}_n.
\end{aligned}$$

So $\mathbf{v}$ is in $\mathrm{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}, \mathbf{v}_{k+1}, \ldots, \mathbf{v}_n\}$. Thus $\mathrm{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}, \mathbf{v}_{k+1}, \ldots, \mathbf{v}_n\} = H$

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Proof of Theorem 5 (cont.)

2. If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is linearly independent, then it is a basis for H. Otherwise, one $\mathbf{v}_k$ is a linear combination of $\{\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}, \mathbf{v}_{k+1}, \ldots, \mathbf{v}_n\}$. In that case $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}, \mathbf{v}_{k+1}, \ldots, \mathbf{v}_n\} = H$.
   If we continue removing vectors from $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ like this, we either get to a subset of $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ which is linearly independent and spans H, and which therefore is a basis for H, or we end up with one vector which spans H. Since $H \neq \{0\}$, the vector in the latter case must be nonzero, and the set consisting of just that one vector is therefore linearly independent, and thus a basis for H.

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Example

Let $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ and $\mathbf{p}_4$ be the elements of $\mathbb{P}$ given by $\mathbf{p}_1(t) = t$, $\mathbf{p}_2(t) = t^2 + t$, $\mathbf{p}_3(t) = -t^2 + 2t$ and $\mathbf{p}_4(t) = -t^2 + 3t$, and let $H = \text{Span}\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4\}$.
Let us find a basis for H.
We have that $\mathbf{p}_4 = \mathbf{p}_1 + \mathbf{p}_3$, so $H = \text{Span}\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$.
$\mathbf{p}_3 = -\mathbf{p}_2 + 3\mathbf{p}_1$, so $H = \text{Span}\{\mathbf{p}_1, \mathbf{p}_2\}$. Since $\mathbf{p}_1 \neq \mathbf{0}$ and $\mathbf{p}_2$ is not a multiple of $\mathbf{p}_1$, the set $\{\mathbf{p}_1, \mathbf{p}_2\}$ is linearly independent, and thus a basis for H.

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The column space of a matrix

Recall that if $A = [\mathbf{a}_1\ \mathbf{a}_2 \ldots \mathbf{a}_n]$ is an $m \times n$ matrix, then $\text{Col}(A) = \text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$.

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Example

Let us find a basis for Col(A) where $A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 1 & -1 \\ 0 & 0 & -2 & 6 \end{bmatrix}$.

We start by reducing A to its reduced echelon form.

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 1 & -1 \\ 0 & 0 & -2 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & -2 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Let $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$ be the columns of A, let B be the reduced echelon form of A, and let $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4$ be the columns of B. The equations $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$ are equivalent (because A and B are row equivalent), so if c_1, c_2, c_3, c_4 are scalars, then $c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + c_3\mathbf{a}_3 + c_4\mathbf{a}_4 = \mathbf{0}$ if and only if $c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + c_3\mathbf{b}_3 + c_4\mathbf{b}_4 = \mathbf{0}$.

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Example (cont.)

It is easy to see that $2\mathbf{b}_1 = \mathbf{b}_2$ and $\mathbf{b}_4 = \mathbf{b}_1 - 3\mathbf{b}_3$. It thus follows that $2\mathbf{a}_1 = \mathbf{a}_2$ and $\mathbf{a}_4 = \mathbf{a}_1 - 3\mathbf{a}_3$. So $\text{Span}\{\mathbf{a}_1, \mathbf{a}_3\} = \text{Col}(A)$.
It is easy to see that the set $\{\mathbf{a}_1, \mathbf{a}_3\}$ is linearly independent.
So $\{\mathbf{a}_1, \mathbf{a}_3\}$ is a basis for $\text{Col}(A)$.

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Bases for the column space of a matrix

Theorem 6

Let A be an $m \times n$ matrix. Then the pivot columns of A form a basis for $\mathrm{Col}(A)$.

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Proof of Theorem 6

Let B be the reduced echelon form of A. Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ be the columns of A, and let $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n$ be the columns of B. Since A and B are row equivalent, the equations $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$ are equivalent, so if $c_1, c_2, \ldots, c_n$ are scalars, then $c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots + c_n\mathbf{a}_n = \mathbf{0}$ if and only if $c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n = \mathbf{0}$. The set of pivot columns of B is linearly independent because all the entries in a pivot column of B are 0 except for one entries which is 1, and these nonzero entries appears in different rows for different columns. It follows that the set of pivot columns of A is linearly independent.

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Proof of Theorem 6 (cont.)

Every nonpivot column of B is a linear combination of the pivot columns of B. It follows that every nonpivot column of A is a linear combination of the pivot columns of A. So the set of pivot columns of A generates $\mathrm{Col}(A)$, and therefore forms a basis for $\mathrm{Col}(A)$.

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The null space of a matrix

Recall that if A is an $m \times n$ matrix, then $\text{Nul}(A) = \{\mathbf{x} : \mathbf{x} \text{ is in } \mathbb{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\}$.

Example

Let us find a basis for Nul(A) where $A = \begin{bmatrix} 1 & -2 & 0 & 3 \\ -2 & 4 & 1 & -9 \\ -1 & 2 & 1 & -6 \end{bmatrix}$.

Nul(A) is the set of solutions to the equation $A\mathbf{x} = \mathbf{0}$. To find a basis for this space, we first reduce the matrix A to its reduced echelon form.

$$\begin{bmatrix} 1 & -2 & 0 & 3 \\ -2 & 4 & 1 & -9 \\ -1 & 2 & 1 & -6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 1 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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Example (cont.)

We see that the elements of Nul(A) are the vectors that can be written in the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_2 - 3x_4 \\ x_2 \\ 3x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ 3 \\ 1 \end{bmatrix}$$

Thus $\text{Nul}(A) = \text{Span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 3 \\ 1 \end{bmatrix} \right\}$.

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Example (cont.)

Since $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 3 \\ 1 \end{bmatrix} \right\}$ is linearly independent, it follows that

$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 3 \\ 1 \end{bmatrix} \right\}$ is a basis for Nul(A).

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Finding a basis for Nul(A)

1. Row reduce A to its reduced echelon form.
2. Write the solutions to the equation $A\mathbf{x} = \mathbf{0}$ as a linear combinations of vectors using the free variables as parameters.
3. These vectors then form a basis for Nul(A).

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The unique representation theorem

Theorem 7

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Then for each $\mathbf{x}$ in V, there exists a unique set of scalars $c_1, c_2, \ldots, c_n$ such that $\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n$.

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Proof of Theorem 7

Since $\mathcal{B}$ spans V, there exist scalars such that $\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n$.
Suppose that also $\mathbf{x} = d_1\mathbf{b}_1 + d_2\mathbf{b}_2 + \cdots + d_n\mathbf{b}_n$ for a set $d_1, d_2, \ldots, d_n$ of scalars. Then

$$\mathbf{0} = \mathbf{x} - \mathbf{x} = (c_1 - d_1)\mathbf{b}_1 + (c_2 - d_2)\mathbf{b}_2 + \cdots + (c_n - d_n)\mathbf{b}_n.$$

Since $\mathcal{B}$ is linearly independent, it follows that $c_1 - d_1 = c_2 - d_2 = \cdots = c_n - d_n = 0$, and thus that $c_1 = d_1, c_2 = d_2, \ldots, c_n = d_n$.

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The coordinates of x relative to the basis $\mathcal{B}$

- Suppose that $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis for a vector space V and that $\mathbf{x}$ is in V. The unique set of scalars $c_1, c_2, \ldots, c_n$ such that $\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n$ is called the *coordinates of* $\boldsymbol{x}$ *relative to the basis* $\mathcal{B}$ (or the $\mathcal{B}$*-coordinates of* x).
- If $c_1, c_2, \ldots, c_n$ are the $\mathcal{B}$-coordinates of x, then the vector $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ is called the *coordinate vector of* $\boldsymbol{x}$ *(relative to* $\mathcal{B}$*)*, or the $\mathcal{B}$*-coordinate vector of* $\boldsymbol{x}$.

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Example

The set $\mathcal{B} = \{1 + t^2, t + t^2, 1 + 2t + t^2\}$ is a basis for $\mathbb{P}_2$. Let us find the coordinate vector of $\mathbf{p}(t) = 3t^2 - 1$.
We must find scalars c_1, c_2, c_3 such that

$$\begin{aligned} 3t^2 - 1 &= c_1(1 + t^2) + c_2(t + t^2) + c_3(1 + 2t + t^2) \\ &= (c_1 + c_2 + c_3)t^2 + (c_2 + 2c_3)t + (c_1 + c_3). \end{aligned}$$

This is equivalent to solving the system

$$\begin{aligned} c_1 + c_2 + c_3 &= 3 \\ c_2 + 2c_3 &= 0 \\ c_1 + c_3 &= -1 \end{aligned}$$

We write up the augmented matrix of the system and reduce it to its reduced echelon form.

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Example (cont.)

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 0 \\ 1 & 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & 0 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & -4 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

It follows that $c_1 = 1$, $c_2 = 4$, $c_3 = -2$, so $[p]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$.

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Example (cont.)

Suppose then that $\mathbf{q}$ is in $\mathbb{P}_2$ and that $[\mathbf{q}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$. Then

$$\mathbf{q} = 1(1 + t^2) + 0(t + t^2) - 3(1 + 2t + t^2) = -2t^2 - 6t - 2.$$

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The coordinate mapping

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. The mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ from V to $\mathbb{R}^n$ is called the *coordinate mapping determined by* $\mathcal{B}$.

Theorem 8

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Then the coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is a one-to-one and onto linear transformation.

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Proof of Theorem 8

Let $\mathbf{u}$ and $\mathbf{v}$ be in V and let c be a scalar. Let $[\mathbf{u}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$

and $[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}$.

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Proof of Theorem 8 (cont.)

Then

$$\begin{aligned}\mathbf{u}+\mathbf{v} &= c_1\mathbf{b}_1+\cdots+c_n\mathbf{b}_n+d_1\mathbf{b}_1+\cdots+d_n\mathbf{b}_n\\ &= (c_1+d_1)\mathbf{b}_1+\cdots+(c_n+d_n)\mathbf{b}_n,\end{aligned}$$

so

$$[\mathbf{u}+\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} c_1+d_1 \\ c_2+d_2 \\ \vdots \\ c_n+d_n \end{bmatrix} = [\mathbf{u}]_{\mathcal{B}}+[\mathbf{v}]_{\mathcal{B}},$$

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Proof of Theorem 8 (cont.)

and

$$c\mathbf{u} = c(c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n) = cc_1\mathbf{b}_1 + \cdots + cc_n\mathbf{b}_n,$$

so

$$[c\mathbf{u}]_{\mathcal{B}} = \begin{bmatrix} cc_1 \\ cc_2 \\ \vdots \\ cc_n \end{bmatrix} = c[\mathbf{u}]_{\mathcal{B}}.$$

This shows that the transformation $\mathbf{u} \mapsto [\mathbf{u}]_{\mathcal{B}}$ is linear.

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Proof of Theorem 8 (cont.)

Suppose that $[\mathbf{u}]_{\mathcal{B}} = \mathbf{0}$. Then $\mathbf{u} = 0\mathbf{b}_1 + 0\mathbf{b}_2 + \cdots + 0\mathbf{b}_n = \mathbf{0}$. This shows that the transformation $\mathbf{u} \mapsto [\mathbf{u}]_{\mathcal{B}}$ is one-to-one.

Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ be in $\mathbb{R}^n$. Let $\mathbf{u} = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \cdots + x_n\mathbf{b}_n$.

Then $[\mathbf{u}]_{\mathcal{B}} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \mathbf{x}$. This shows that the transformation $\mathbf{u} \mapsto [\mathbf{u}]_{\mathcal{B}}$ is onto.

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Example

Let us use coordinate vectors to determine if the set $\{1-t^2, t-t^2, 2-t+t^2\}$ is a basis for $\mathbb{P}_2$.
Let $\mathcal{B} = \{1, t, t^2\}$ be the standard basis for $\mathbb{P}_2$. Then

$$[1-t^2]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad [t-t^2]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \quad [2-t+t^2]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$.

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Example (cont.)

Then

$$\det(A) = \begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ -1 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} + 2 \begin{vmatrix} 0 & 1 \\ -1 & -1 \end{vmatrix}$$
$$= 1 - 1 + 2(0 + 1) = 2$$

so $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$.

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Example (cont.)

Let $\mathbf{p}$ be in $\mathbb{P}_2$. Then there are scalars c_1, c_2, c_3 such that

$$
\begin{aligned}
[\mathbf{p}]_{\mathcal{B}} &= c_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \\
&= c_1[1 - t^2]_{\mathcal{B}} + c_2[t - t^2]_{\mathcal{B}} + c_3[2 - t + t^2]_{\mathcal{B}} \\
&= [c_1(1 - t^2) + c_2(t - t^2) + c_3(2 - t + t^2)]_{\mathcal{B}}.
\end{aligned}
$$

Since the coordinate mapping it one-to-one, it follows that $\mathbf{p} = c_1(1 - t^2) + c_2(t - t^2) + c_3(2 - t + t^2)$. Thus, $\mathbf{p}$ is in $\text{Span}\{1 - t^2, t - t^2, 2 - t + t^2\}$. This shows that $\text{Span}\{1 - t^2, t - t^2, 2 - t + t^2\} = \mathbb{P}_2$.

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Example (cont.)

Suppose that $c_1(1-t^2)+c_2(t-t^2)+c_3(2-t+t^2)=\mathbf{0}$. Then

$$c_1\begin{bmatrix}1\\0\\-1\end{bmatrix}+c_2\begin{bmatrix}0\\1\\-1\end{bmatrix}+c_3\begin{bmatrix}2\\-1\\1\end{bmatrix}=$$

$$c_1[1-t^2]_{\mathcal{B}}+c_2[t-t^2]_{\mathcal{B}}+c_3[2-t+t^2]_{\mathcal{B}}=$$

$$[c_1(1-t^2)+c_2(t-t^2)+c_3(2-t+t^2)]_{\mathcal{B}}=[\mathbf{0}]_{\mathcal{B}}=\mathbf{0}.$$

Since $\left\{\begin{bmatrix}1\\0\\-1\end{bmatrix},\begin{bmatrix}0\\1\\-1\end{bmatrix},\begin{bmatrix}2\\-1\\1\end{bmatrix}\right\}$ is linearly independent, it follows that $c_1=c_2=c_3=0$.

This shows that $\{1-t^2,t-t^2,2-t+t^2\}$ is linearly independent, and thus a basis for $\mathbb{P}_2$.

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Example

Let $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, and $\mathbf{x} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$.

Then $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ is a basis for $\mathbb{R}^2$. Let us find the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$.

We must find c_1 and c_2 such that $\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2$. Let $P_{\mathcal{B}} = [\mathbf{b}_1\ \mathbf{b}_2] = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$. If $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, then

$\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 = P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}$.

$\det(P_{\mathcal{B}}) = 1$, so $P_{\mathcal{B}}$ is invertible. It follows that $[\mathbf{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}\mathbf{x}$.

$P_{\mathcal{B}}^{-1} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$, so $[\mathbf{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}\mathbf{x} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

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Coordinates in $\mathbb{R}^n$

- Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ be a basis for $\mathbb{R}^n$.
- Let $P_{\mathcal{B}} = [\mathbf{b}_1\ \mathbf{b}_2 \ldots \mathbf{b}_n]$. It follows from the inverse matrix theorem that P_B is invertible.
- If $\mathbf{x}$ is in $\mathbb{R}^n$, then $\mathbf{x} = P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}$, from which it follows that $[\mathbf{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}\mathbf{x}$.

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Example

Suppose we have a large container that contains two chemical substances, substance A and substance B. Substance A contains 3 g sulphuric acid and 1 g acetic acid per 100 g, and substance B contains 5 g sulphuric acid and 2 g acetic acid per 100 g. How much of substance A and how much how substance B must we add or remove from the container in order to increase the amount of sulphuric acid in the container by 2 g and reduce the amount of acetic acid by 1 g?

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Solution

Suppose we add $x_1 \cdot$ 100 g of substance A and $x_2 \cdot$ 100 g of substance B. The we must have that

$$x_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

We have that $\det \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} = 1$, so $\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$ is invertible and $\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \end{bmatrix} \right\}$ is therefore a basis for $\mathbb{R}^2$.

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Solution (cont.)

So

$$x_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

if and only if

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 9 \\ -5 \end{bmatrix}.$$

So we must add 900 g of substance A and remove 500 g of substance B in order to increase the amount of sulphuric acid in the container by 2 g and reduce the amount of acetic acid by 1 g.

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Solution (cont.)

In general, if we want to increase the amount of sulphuric acid in the container by a g and reduce the amount of acetic acid by b g, then we must add $x_1 \cdot 100$ g of substance A and $x_2 \cdot 100$ g of substance B where

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} a \\ -b \end{bmatrix} = \begin{bmatrix} 2a - 5b \\ -a + 3b \end{bmatrix}.$$

Plan for next week

Wednesday we shall introduce and study

- the *dimension* of a vector space,
- the *rank* of a matrix.

Sections 4.5–4.6 in “Linear Algebras and Its Applications” (pages 225–238).

Thursday we shall look at applications to *Markov chains*. Section 4.9 in “Linear Algebras and Its Applications” (pages 253–264).

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