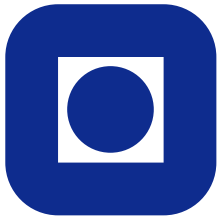

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TMA4115 - Calculus 3

Lecture 17, March 13

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Norwegian University of Science and Technology

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Review of last week's lecture

Last week we looked at

- the *determinate* of a square matrix,
- *Cramer's rule*,
- a formula for the inverse of an invertible matrix,
- the relationship between areas, volumes and determinants.

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Today's lecture

Today we shall introduce and study

- abstract *vector spaces* and *subspaces*,
- *null spaces* and *column spaces* of matrices,
- *linear transformations* between abstract vector spaces,
- *kernels* and *ranges* of linear transformations,
- *linear independence* and *linear dependence* in abstract vector spaces.

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Vector spaces

Definition

A *vector space* is a nonempty set V of objects, called vectors, on which are defined two operations, called addition and multiplication by scalars (real numbers), subject to the ten axioms listed on the next slide.

1. The *sum* of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u}+\mathbf{v}$, is in V.
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.
3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.
4. There is a *zero* vector $\mathbf{0}$ in V such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$.
5. For each $\mathbf{u}$ in V, there is a vector $-\mathbf{u}$ in V such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.
6. The *scalar multiple* of $\mathbf{u}$ by c, denoted by $c\mathbf{u}$, is in V.
7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$.
8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$.
9. $c(d\mathbf{u})=(cd)\mathbf{u}$.
10. $1\mathbf{u}=\mathbf{u}$.

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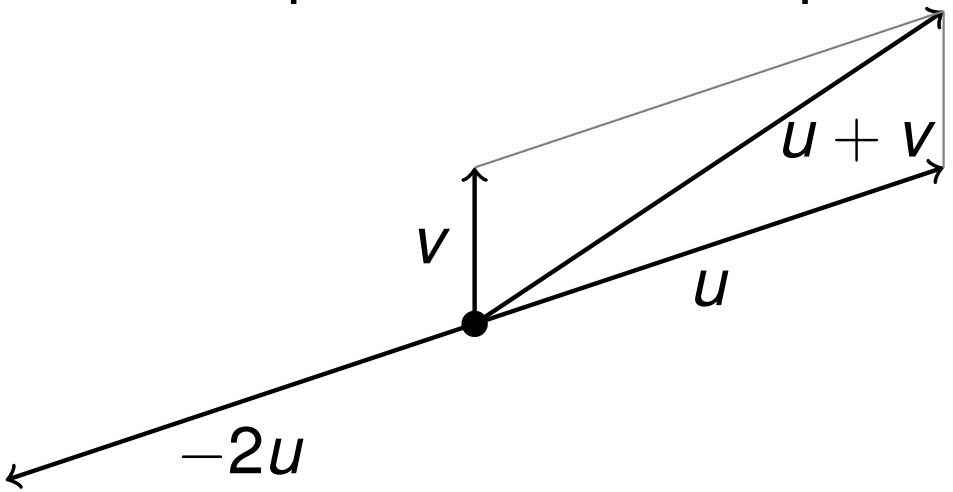
Examples of vector spaces

1. Let n be a positive integer. Then $\mathbb{R}^n$ is a vector space.
2. The set of arrows in a fixed plane (or space), starting at one fixed point is a vector space.



3. Let m and n be positive integers. Then the set $M_{m\times n}$ of all $m \times n$ matrices is a vector space.

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Examples of vector spaces

4. Let $\mathbb{S}$ be the set of all doubly infinite sequences of numbers: $\{y_k\}_{k\in\mathbb{Z}}$. If $\{z_k\}_{k\in\mathbb{Z}}$ is another element of $\mathbb{S}$, then the sum $\{y_k\}_{k\in\mathbb{Z}} + \{z_k\}_{k\in\mathbb{Z}}$ is sequence $\{y_k + z_k\}_{k\in\mathbb{Z}}$, and if c is a scalar, then $c\{y_k\}_{k\in\mathbb{Z}}$ is the sequence $\{cy_k\}_{k\in\mathbb{Z}}$. Then $\mathbb{S}$ is a vector space.
5. Let I be an interval and let V be the space of all real-valued functions defined on I. If f, g are in V and c is a scalar, then $f + g$ is the function defined by $(f + g)(x) = f(x) + g(x)$ for x in I, and cf is the function defined by $(cf)(x) = cf(x)$ for x in I. Then V is a vector space.

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Subspaces

Definition

A *subspace* of a vector space V is a subset H of V that has 3 properties:

1. The zero vector of V is in H.
2. If $\mathbf{u}$ and $\mathbf{v}$ are in H, then $\mathbf{u} + \mathbf{v}$ is in H.
3. If $\mathbf{u}$ is in H and c is a scalar, then $c\mathbf{u}$ is in H.

Every subspace is a vector space. Conversely, every vector space is a subspace (of itself and possibly of other larger spaces).

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Examples of subspaces

Let V be the space of all real-valued functions defined on $\mathbb{R}$.

1. Let $\mathbb{P}$ be the set of all polynomials with real coefficients. Then $\mathbb{P}$ is a subspace of V.
2. For each $n \geq 0$, let $\mathbb{P}_n$ be the set of polynomials with real coefficients of degree at most n. Then $\mathbb{P}_n$ is a subspace of $\mathbb{P}$ (and of V).
3. Let $C(\mathbb{R})$ be the set of all real-valued continuous functions defined on $\mathbb{R}$. Then $C(\mathbb{R})$ is a subspace of V.

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The zero subspace

Let V be a vector space, and let $\mathbf{0}$ be the zero vector of V. The set consisting of only $\mathbf{0}$ is a subspace of V. This subspace is called the *zero subspace* and is written as $\{\mathbf{0}\}$.

Subspaces spanned by sets

Let V be a vector space.

1. If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are vectors in V and $c_1, c_2, \ldots, c_n$ are scalars, then the vector $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$ is called a *linear combination* of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.
2. We let $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ denote the set of all possible linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.
3. $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a subspace of V.
4. $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is called the *subspace spanned* (or *generated*) by $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.
5. Given any subspace H of V, a *spanning* (or *generating*) set for H is a set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ in H such that $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$.

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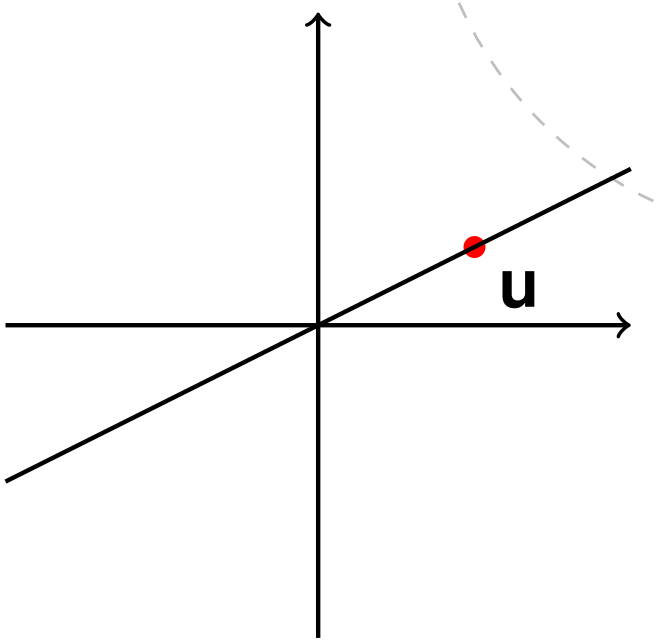
Example

Let $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. The subspace generated by $\mathbf{u}$ is the line which goes through $\mathbf{0}$ and $\mathbf{u}$.



Let $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Since every vector in $\mathbb{R}^2$ can be written as a linear combination of $\mathbf{u}$ and $\mathbf{v}$, it follows that subspace generated by $\mathbf{u}$ and $\mathbf{v}$ is $\mathbb{R}^2$.

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Example

Let H be the set of all vectors of the form $\begin{bmatrix} 2a-3b \\ 0 \\ -a+b \end{bmatrix}$

where a and b represent arbitrary real numbers.

Let $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$. Then

$$a\mathbf{v}_1 + b\mathbf{v}_2 = \begin{bmatrix} 2a-3b \\ 0 \\ -a+b \end{bmatrix},$$

so $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. It follows that H is a subspace of $\mathbb{R}^3$.

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Example

Let H be the set of all vectors of the form $\begin{bmatrix} 4a+3b \\ -1 \\ -a+2b \end{bmatrix}$

where a and b represent arbitrary real numbers.
Then H is not a subspace of $\mathbb{R}^3$ because $\mathbf{0}$ is not in H.

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The null space of a matrix

Let A be an $m \times n$ matrix.

1. The *null space* of A, written as $\text{Nul}(A)$, is the set of all solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.
2. $\text{Nul}(A) = \{\mathbf{x} : \mathbf{x} \text{ is in } \mathbb{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\}$.
3. $\text{Nul}(A)$ is a subspace of $\mathbb{R}^n$ because:
   - $\mathbf{0} \in \text{Nul}(A)$ since $A\mathbf{0} = \mathbf{0}$,
   - if $\mathbf{x}, \mathbf{y} \in \text{Nul}(A)$, then $\mathbf{x} + \mathbf{y} \in \text{Nul}(A)$ since $A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \mathbf{0} + \mathbf{0} = \mathbf{0}$,
   - if $\mathbf{x} \in \text{Nul}(A)$ and c is a scalar, then $c\mathbf{x} \in \text{Nul}(A)$ since $A(c\mathbf{x}) = cA\mathbf{x} = c\mathbf{0} = \mathbf{0}$.

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Example

Let $A = \begin{bmatrix} 3 & 4 & 0 \\ -1 & 2 & 2 \end{bmatrix}$ and let $\mathbf{x} = \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix}$.

Then $\mathbf{x}$ is not in Nul(A) because

$$A\mathbf{x} = \begin{bmatrix} 3 & 4 & 0 \\ -1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -10 \end{bmatrix} \neq \mathbf{0}.$$

Let us find an explicit description of Nul(A) by listing vectors that span Nul(A). Nul(A) is the set of all solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$. To find the solution set of $A\mathbf{x} = \mathbf{0}$ we reduce the augmented matrix $[A\ \mathbf{0}]$ to its reduced echelon form.

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Example (cont.)

$$\begin{bmatrix} 3 & 4 & 0 & 0 \\ -1 & 2 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} -1 & 2 & 2 & 0 \\ 3 & 4 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} -1 & 2 & 2 & 0 \\ 0 & 10 & 6 & 0 \end{bmatrix} \to$$

$$\begin{bmatrix} -1 & 2 & 2 & 0 \\ 0 & 1 & 0.6 & 0 \end{bmatrix} \to \begin{bmatrix} -1 & 0 & 0.8 & 0 \\ 0 & 1 & 0.6 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -0.8 & 0 \\ 0 & 1 & 0.6 & 0 \end{bmatrix}$$

We see that the solution set of $A\mathbf{x} = \mathbf{0}$ is the set of vectors which can be written on the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.8x_3 \\ -0.6x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0.8 \\ -0.6 \\ 1 \end{bmatrix}$$ where x_3 is a free parameter.

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Example (cont.)

$$\text{Thus, } \text{Nul}(A) = \text{Span}\left\{ \begin{bmatrix} 0.8 \\ -0.6 \\ 1 \end{bmatrix} \right\}.$$

Example

Let H be the set of vectors $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ which satisfy that

$2a + 2b = c$ and $a + b + c = 3d$.

Then $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ is in H if and only if

$2a + 2b - c = 0$ and $a + b + c - 3d = 0$.

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Example (cont.)

So if $A = \begin{bmatrix} 2 & 2 & -1 & 0 \\ 1 & 1 & 1 & -3 \end{bmatrix}$, then $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ is in H if and only if it is in $\text{Nul}(A)$.

Thus, $H = \text{Nul}(A)$, and H is therefore a subspace of $\mathbb{R}^4$, and hence a vector space.

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The column space of a matrix

Let A be an $m \times n$ matrix.

1. The *column space* of A, written as $\text{Col}(A)$, is the set of all linear combinations of the columns of A.
2. If $A = [\mathbf{a}_1\ \mathbf{a}_2 \ldots \mathbf{a}_n]$, then $\text{Col}(A) = \text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$.
3. $\text{Col}(A) = \{\mathbf{b} : \mathbf{b} = A\mathbf{x} \text{ for some } \mathbf{x} \text{ in } \mathbb{R}^n\}$.
4. $\text{Col}(A)$ is a subspace of $\mathbb{R}^m$.

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Example

Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 0 & 2 \\ 0 & 3 & -3 \end{bmatrix}$.

Then $\text{Col}(A) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix} \right\}$.

Let $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, and let us determine if $\mathbf{b}$ is in $\text{Col}(A)$. $\mathbf{b}$ is in $\text{Col}(A)$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ is consistent. To determine if the equation $A\mathbf{x} = \mathbf{b}$ is consistent, we reduce the augmented matrix $[A\ \mathbf{b}]$ to an echelon form.

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Example (cont.)

$$\begin{bmatrix} 1 & -2 & 3 & 1 \\ 2 & 0 & 2 & 2 \\ 0 & 3 & -3 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 4 & -4 & 0 \\ 0 & 3 & -3 & 3 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We see that the equation $A\mathbf{x} = \mathbf{b}$ is inconsistent, so $\mathbf{b}$ is not in Col(A).

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Example

Let H be the set of vectors of the form $\begin{bmatrix} 2r-s \\ r \\ -3s \end{bmatrix}$ where r and s represent arbitrary real numbers. Let us determine if H is a vector space.

Let $A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \\ 0 & -3 \end{bmatrix}$.

Then $\text{Col}(A) = H$, so H is a subspace of $\mathbb{R}^3$ and is therefore a vector space.

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Linear transformations

Definition

A *linear transformation* T from a vector space V into a vector space W is a rule that assigns to each vector $\mathbf{x}$ in V a unique vector $T(\mathbf{x})$ in W such that

1. $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in V,
2. $T(c\mathbf{v}) = cT(\mathbf{v})$ for all $\mathbf{v}$ in V all scalars c.

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Example

- Let V be the set of all real-valued functions f defined on an interval $[a, b]$ with the property that they are differentiable and their derivatives are continuous functions on $[a, b]$.
- Let $W = C[a, b]$ be the set of real-valued continuous functions on $[a, b]$.
- Then V and W are vector spaces.
- Let $D : V \to W$ be the transformation that maps f in V to its derivative f'.
- Then $D(f + g) = (f + g)' = f' + g' = D(f) + D(g)$ and $D(cf) = (cf)' = cf' = cD(f)$, so D is a linear transformation.

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The kernel and range of a linear transformation

Let V and W be vector spaces and let T be a linear transformation from V to W.

1. The *kernel* (or *null space*) of T is the set $\{\mathbf{u} : \mathbf{u} \text{ is in } V \text{ and } T(\mathbf{v}) = \mathbf{0}\}$ of all $\mathbf{u}$ in V for which $T(\mathbf{u}) = \mathbf{0}$.
2. The *range* of T is the set $\{\mathbf{w} : \mathbf{w} = T(\mathbf{v}) \text{ for some } \mathbf{v} \text{ in } V\}$ of all vectors in W of the form $T(\mathbf{v})$ for some $\mathbf{v}$ in V.
3. The kernel of T is a subspace of V.
4. The range of T is a subspace of W.
5. If $V = \mathbb{R}^n$, $W = \mathbb{R}^m$, and A is the standard matrix of T, then the kernel of T is the null space of A, and the range of T is the column space of A.

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Example

- Let $y'' + p(t)y' + q(t)y = 0$ be a homogenous second-order linear differential equation.
- Let V be the set of all real-valued twice continuously differentiable functions defined on $\mathbb{R}$.
- Let $W = C(\mathbb{R})$ be the set of real-valued continuous functions on $\mathbb{R}$.
- Let $T : V \to W$ be the transformation that maps f in V to $f'' + p(t)f' + q(t)f$.
- Then V and W are vector spaces, and T is a linear transformation.
- A function f in V is in the kernel of T if and only if f is a solution to the differential equation $y'' + p(t)y' + q(t)y = 0$.

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Example

Define $T : \mathbb{P} \to \mathbb{R}$ by $T(\mathbf{p}) = \mathbf{p}(1)$.
If $\mathbf{p}$ and $\mathbf{q}$ are in $\mathbb{P}$ and c is a scalar, then

$$T(\mathbf{p} + \mathbf{q}) = (\mathbf{p} + \mathbf{q})(1) = \mathbf{p}(1) + \mathbf{q}(1) = T(\mathbf{p}) + T(\mathbf{q})$$

and

$$T(c\mathbf{p}+) = (c\mathbf{p})(1) = c\mathbf{p}(1) = cT(\mathbf{p}),$$

so T is linear.
$\mathbf{p}$ is in the kernel of T if and only if $\mathbf{p}(1) = 0$, so the kernel of T is the set $\{(x-1)\mathbf{q} : \mathbf{q} \text{ is in } \mathbb{P}\}$.
If r is a real number, then r is also a (constant) polynomial, so r is in $\mathbf{p}$ and $T(r) = r(1) = r$. This shows that the range of T is $\mathbb{R}$ (so T is onto).

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Linear independence

Let V be a vector space, and let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be an indexed set of vectors in V.

- $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is said to be *linearly independent* if the vector equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$ has only the trivial solution.
- $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is said to be *linearly dependent* if there exist scalars $c_1, c_2, \ldots, c_n$ not all equal to zero such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$.

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Linear independence in $\mathbb{R}^n$

- Recall that if $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ are vectors in $\mathbb{R}^n$, then the indexed set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is linearly independent if and only if the matrix $[\mathbf{v}_1\ \mathbf{v}_2 \ldots \mathbf{v}_p]$ has a pivot position in every column.
- There is no similar method to determine if a indexed set of vectors in an arbitrary vector space is linearly independent.

Example

Let $\mathbf{p}_1$, $\mathbf{p}_2$ and $\mathbf{p}_3$ be the elements of $\mathbb{P}$ (the vector space of polynomials with real coefficients) given by $\mathbf{p}_1(t) = 1$, $\mathbf{p}_2(t) = t - 1$ and $\mathbf{p}_3(t) = t^2 + t$.
Let us show that $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is linearly independent.
Assume that $c_1\mathbf{p}_1 + c_2\mathbf{p}_2 + c_3\mathbf{p}_3 = \mathbf{0}$. By evaluating at $t = 0$, $t = 1$, $t = 2$, we get

$$
\begin{aligned}
0 &= c_1\mathbf{p}_1(0) + c_2\mathbf{p}_2(0) + c_3\mathbf{p}_3(0) = c_1 - c_2 \\
0 &= c_1\mathbf{p}_1(1) + c_2\mathbf{p}_2(1) + c_3\mathbf{p}_3(1) = c_1 + 2c_3 \\
0 &= c_1\mathbf{p}_1(2) + c_2\mathbf{p}_2(2) + c_3\mathbf{p}_3(2) = c_1 + c_2 + 6c_3
\end{aligned}
$$

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Example (cont.)

Let $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 2 \\ 1 & 1 & 6 \end{bmatrix}$. Then

$$\det(A) = \begin{vmatrix} 1 & -1 & 0 \\ 1 & 0 & 2 \\ 1 & 1 & 6 \end{vmatrix} = \begin{vmatrix} 0 & 2 \\ 1 & 6 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 1 & 6 \end{vmatrix} = -2 + 6 - 2 = 2 \neq 0$$

So A is invertible. It follows that the only solution to the system

$$\begin{aligned} 0 &= c_1 - c_2 \\ 0 &= c_1 + 2c_3 \\ 0 &= c_1 + c_2 + 6c_3 \end{aligned}$$

is $c_1 = c_2 = c_3 = 0$.

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Example (cont.)

So the only solution to the equation $c_1\mathbf{p}_1 + c_2\mathbf{p}_2 + c_3\mathbf{p}_3 = \mathbf{0}$ is $c_1 = c_2 = c_3 = 0$. This shows that $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is linearly independent.

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Linear dependence

Theorem 4

An indexed set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ of two or more vectors in a vector set V is linearly dependent if and only if $\mathbf{v}_1 = \mathbf{0}$ or some $\mathbf{v}_j$, with $j > 1$, is a linear combination of the preceding vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{j-1}$.

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Example

Let $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ and $\mathbf{p}_4$ be the elements of $\mathbb{P}$ (the vector space of polynomials with real coefficients) given by $\mathbf{p}_1(t) = 1$, $\mathbf{p}_2(t) = t$, $\mathbf{p}_3(t) = t^2 - 2$ and $\mathbf{p}_4(t) = t^2 + 2t - 3$.
Then $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4\}$ is linearly dependent because $\mathbf{p}_4 = \mathbf{p}_3 + 2\mathbf{p}_2 - \mathbf{p}_1$.

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Tomorrow's lecture

Tomorrow we shall introduce and study

- *bases* of vector spaces,
- *coordinate systems* in vector spaces relative to bases.

Section 4.3–4.4 in "Linear Algebras and Its Applications" (pages 208–225).

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