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TMA4115 - Calculus 3

Lecture 16, March 7

Toke Meier Carlsen
Norwegian University of Science and Technology

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Yesterday’s lecture

Yesterday we introduce and study *determinants*.

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Today's lecture

Today we shall

- look at *Cramer's rule*,
- give a formula for the inverse of an invertible matrix,
- look at the relationship between areas, volumes and determinants.

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The inverse of an invertible 2×2 matrix

Theorem 4

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $\det(A) \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

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Cofactor expansions

When $A = [a_{ij}]$, the (i, j)-cofactor of A is the number $C_{ij} = (-1)^{i+j} \det(A_{ij})$.

Theorem 1

Let $A = [a_{ij}]$ be an $n \times n$ matrix. Then

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

and

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

for any i and any j between 1 and n.

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The determinant of a triangular matrix

A *triangular* matrix is a square matrix $A = [a_{ij}]$ for which $a_{ij} = 0$ when $i > j$.

Theorem 2

If A is a triangular matrix, then $\det(A)$ is the product of the entries on the main diagonal of A.

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Properties of determinants

Theorem 3

Let A be a square matrix.

1. If a multiple of one row of A is added to another row to produce a matrix B, then $\det(B) = \det(A)$.
2. If two rows of A are interchanged to produce B, then $\det(B) = -\det(A)$.
3. If one row of A is multiplied by k to produce B, then $\det(B) = k\det(A)$.

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Properties of determinants

Theorem 4

A square matrix A is invertible if and only if $\det(A) \neq 0$.

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Column operations

Theorem 5

If A is a square matrix, then $\det(A^T) = \det(A)$.

Multiplicative property

Theorem 6

If A and B are $n \times n$ matrices, then $\det(AB) = \det(A)\det(B)$.

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Cramer's rule

For any $n \times n$ matrix A and any $\mathbf{b}$ in $\mathbb{R}^n$, let $A_i(\mathbf{b})$ be the matrix obtained from A by replacing column i by the vector $\mathbf{b}$.

Theorem 7

Let A be an invertible $n \times n$ matrix. For any $\mathbf{b}$ in $\mathbb{R}^n$, the unique solution $\mathbf{x}$ of the equation $A\mathbf{x} = \mathbf{b}$ has entries given by

$$x_i = \frac{\det(A_i(\mathbf{b}))}{\det(A)} \text{ for } i = 1, 2, \ldots, n.$$

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Proof of Theorem 7

Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ be the columns of A, and let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ be the columns of I_n. If $A\mathbf{x} = \mathbf{b}$, then

$$\begin{aligned} A(I_n)_i(\mathbf{x}) &= A[\mathbf{e}_1 \ldots \mathbf{e}_{i-1}\ \mathbf{x}\ \mathbf{e}_{i+1} \ldots \mathbf{e}_n] \\ &= [A\mathbf{e}_1 \ldots A\mathbf{e}_{i-1}\ A\mathbf{x}\ A\mathbf{e}_{i+1} \ldots A\mathbf{e}_n] \\ &= [\mathbf{a}_1 \ldots \mathbf{a}_{i-1}\ \mathbf{b}\ \mathbf{a}_{i+1} \ldots \mathbf{a}_n] = A_i(\mathbf{b}), \end{aligned}$$

so $\det(A_i(\mathbf{b})) = \det(A(I_n)_i(\mathbf{x})) = \det(A)\det((I_n)_i(\mathbf{x}))$.
By using cofactor expansion along row i, we see that $\det((I_n)_i(\mathbf{x})) = (-1)^{i+i}x_i \det(I_{n-1}) = x_i$, so $\det(A_i(\mathbf{b})) = x_i \det(A)$. Since A is invertible, $\det(A) \neq 0$, so $x_i = \frac{\det(A_i(\mathbf{b}))}{\det(A)}$.

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Example

Let us use Cramer's rule to solve the linear system

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 4 \\ -x_1 + 2x_3 &= 2 \\ 3x_1 + x_2 + 3x_3 &= -2 \end{aligned}$$

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Solution

Let $A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & 2 \\ 3 & 1 & 3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix}$. Then the system is equivalent to the matrix equation $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{b}$.

$$\det(A) = \begin{vmatrix} 2 & 1 & 1 \\ -1 & 0 & 2 \\ 3 & 1 & 3 \end{vmatrix} = -(-1) \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} - 2 \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} = 2 + 2 = 4,$$

so A is invertible. We can therefore use Cramer's rule to find the unique solution of the system.

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Solution (cont.)

$$\det(A_1(\mathbf{b})) = \begin{vmatrix} 4 & 1 & 1 \\ 2 & 0 & 2 \\ -2 & 1 & 3 \end{vmatrix} = -2\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} - 2\begin{vmatrix} 4 & 1 \\ -2 & 1 \end{vmatrix}$$

$$= -4 - 12 = -16$$

$$\det(A_2(\mathbf{b})) = \begin{vmatrix} 2 & 4 & 1 \\ -1 & 2 & 2 \\ 3 & -2 & 3 \end{vmatrix}$$

$$= -(-1)\begin{vmatrix} 4 & 1 \\ -2 & 3 \end{vmatrix} + 2\begin{vmatrix} 2 & 1 \\ 3 & 3 \end{vmatrix} - 2\begin{vmatrix} 2 & 4 \\ 3 & -2 \end{vmatrix}$$

$$= 14 + 6 + 32 = 52$$

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Solution (cont.)

$$\det(A_3(\mathbf{b})) = \begin{vmatrix} 2 & 1 & 4 \\ -1 & 0 & 2 \\ 3 & 1 & -2 \end{vmatrix} = -(-1)\begin{vmatrix} 1 & 4 \\ 1 & -2 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix}$$
$$= -6 + 2 = -4$$

So $x_1 = \frac{-16}{4} = -4$, $x_2 = \frac{52}{4} = 13$, $x_3 = \frac{-4}{4} = -1$ is the unique solution to the system.

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Example

Let us find the values of the parameter s for which the system

$$\begin{aligned} 2sx_1 + x_2 &= 1 \\ 3sx_1 + 6sx_2 &= 2 \end{aligned}$$

has a unique solution, and then find this solution.

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Solution

Let $A = \begin{bmatrix} 2s & 1 \\ 3s & 6s \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.
$\det(A) = 12s^2 - 3s = 3s(4s-1) \neq 0$ if and only if $s \neq 0$ and $s \neq 1/4$, so the system has a unique solution unless $s = 0$ or $s = 1/4$.

$$\det(A_1(\mathbf{b})) = \begin{vmatrix} 1 & 1 \\ 2 & 6s \end{vmatrix} = 6s - 2$$

$$\det(A_2(\mathbf{b})) = \begin{vmatrix} 2s & 1 \\ 3s & 2 \end{vmatrix} = 4s - 3s = s$$

So $x_1 \frac{6s-2}{12s^2-3s}$, $x_2 = \frac{s}{12S^2-3s}$ is the unique solution to the system when $s \neq 0$ and $s \neq 1/4$.

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An inverse formula

When A is an $n \times n$ matrix, then $\operatorname{adj}(A)$ is the $n \times n$ matrix whose (i, j)-entry is $C_{ji} = (-1)^{i+j} \det(A_{ji})$.

Theorem 8

Let A be an invertible $n \times n$ matrix. Then

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A).$$

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Proof of Theorem 8

Let A be an invertible $n \times n$ matrix. Then $A(A^{-1}\mathbf{e}_j) = (AA^{-1})\mathbf{e}_j = I_n\mathbf{e}_j = \mathbf{e}_j$, so the jth column of A^{-1} is the unique solution to the equation $A\mathbf{x} = \mathbf{e}_j$.

It follows that the (i,j)-entry is $\frac{\det(A_i(\mathbf{e}_j))}{\det(A)}$. By using cofactor expansion along the jth column, we see that $\det(A_i(\mathbf{e}_j)) = (-1)^{i+j}\det(A_{ji}) = C_{ji}$, so $A^{-1} = \frac{1}{\det(A)}\,\mathrm{adj}(A)$.

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Example

Let us find the inverse of the matrix $A = \begin{bmatrix} 3 & 6 & 7 \\ 0 & 2 & 1 \\ 2 & 3 & 4 \end{bmatrix}$

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Solution

$$\det(A) = \begin{vmatrix} 3 & 6 & 7 \\ 0 & 2 & 1 \\ 2 & 3 & 4 \end{vmatrix} = 2 \begin{vmatrix} 3 & 7 \\ 2 & 4 \end{vmatrix} - \begin{vmatrix} 3 & 6 \\ 2 & 3 \end{vmatrix} = -4 + 3 = -1$$

so A is invertible.

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Solution (cont.)

$$C_{11} = \begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = 5 \quad C_{21} = -\begin{vmatrix} 6 & 7 \\ 3 & 4 \end{vmatrix} = -3 \quad C_{31} = \begin{vmatrix} 6 & 7 \\ 2 & 1 \end{vmatrix} = -8$$

$$C_{12} = -\begin{vmatrix} 0 & 1 \\ 2 & 4 \end{vmatrix} = 2 \quad C_{22} = \begin{vmatrix} 3 & 7 \\ 2 & 4 \end{vmatrix} = -2 \quad C_{32} = -\begin{vmatrix} 3 & 7 \\ 0 & 1 \end{vmatrix} = -3$$

$$C_{13} = \begin{vmatrix} 0 & 2 \\ 2 & 3 \end{vmatrix} = -4 \quad C_{23} = -\begin{vmatrix} 3 & 6 \\ 2 & 3 \end{vmatrix} = 3 \quad C_{33} = \begin{vmatrix} 3 & 6 \\ 0 & 2 \end{vmatrix} = 6$$

Solution (cont.)

So $\operatorname{adj}(A) = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = \begin{bmatrix} 5 & -3 & -8 \\ 2 & -2 & -3 \\ -4 & 3 & 6 \end{bmatrix}$ and

$\det(A) = \frac{1}{\det(A)} \operatorname{adj}(A) = \begin{bmatrix} -5 & 3 & 8 \\ -2 & 2 & 3 \\ 4 & -3 & -6 \end{bmatrix}$.

Determinants as area or volume

Theorem 9

If A is a 2×2 matrix, then the area of the parallelogram determined by the columns of A is $|\det(A)|$.
If A is a 3×3 matrix, then the area of the parallelepiped determined by the columns of A is $|\det(A)|$.

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Example

Let us find the area of the parallelogram whose vertices are $(-2, 0)$, $(-3, 3)$, $(2, -5)$ and $(1, -2)$.

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Solution

By translating by $(2, 0)$ we see that the area of the parallelogram is the same as the area of the parallelogram whose vertices are $(0, 0)$, $(-1, 3)$, $(4, -5)$ and $(3, -2)$. Thus the area of the parallelogram is

$$\left|\det \begin{bmatrix} -1 & 4 \\ 3 & -5 \end{bmatrix}\right| = |-7| = 7.$$

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Areas and linear transformations

If T is a transformation and S is a set in the domain of T, then we let $T(S)$ denote the set of images of points in S.

Theorem 10

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation determined by a 2×2 matrix A. If S is a region in $\mathbb{R}^2$ with finite area, then area of $T(S) = |\det(A)|(\text{area of } S)$.
Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation determined by a 3×3 matrix A. If S is a region in $\mathbb{R}^2$ with finite volume, then volume of $T(S) = |\det(A)|(\text{volume of } S)$.

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Solution

Observe that a point (x_1, x_2) is in E if and only if $\frac{x_1^2}{a_2} + \frac{x_2^2}{b^2} \leq 1$. Let D be the unit disk, i.e., the set of points (x_1, x_2) which satisfy that $x_1^2 + x_2^2 \leq 1$. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation $(x_1, x_2) \mapsto (ax_1, bx_2)$. Then $T(D) = E$ because if $y_1 = ax_1$ and $y_2 = bx_2$, then $\frac{y_1^2}{a^2} + \frac{y_2^2}{b^2} = x_1^2 + x_2^2$. So

$$\text{area of } E = \text{area of } D \cdot \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = \pi ab.$$

Problem 5 from the exam from August 2011

Let $A = \begin{bmatrix} 1 & 2 & a \\ 3 & 7 & b \\ 2 & 9 & c \end{bmatrix}$.

1. Decide for which values of a, b and c, the matrix A is invertible.
2. Find values of a, b and c for which A^{-1} is an integer matrix.

Solution

$$\det(A) = \begin{vmatrix} 1 & 2 & a \\ 3 & 7 & b \\ 2 & 9 & c \end{vmatrix} = a \begin{vmatrix} 3 & 7 \\ 2 & 9 \end{vmatrix} - b \begin{vmatrix} 1 & 2 \\ 2 & 9 \end{vmatrix} + c \begin{vmatrix} 1 & 2 \\ 3 & 7 \end{vmatrix} = 13a - 5b + c$$

So A is invertible if and only if $13a - 5b + c \neq 0$.
If $13a - 5b + c \neq 0$, then

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) = \frac{1}{13a - 5b + c} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}.$$

If a, b and c are integers, then each cofactor C_{ji} is an integer (because the determinant of an integer matrix is an integer), so A^{-1} is an integer matrix if a, b and c are integers and $13a - 5b + c = \pm 1$. This is, for instance, the case if $a = b = 0$ and $c = 1$.

Problem 4 from August 2007

1. A square 3×3 matrix A is given by

$$A = \begin{bmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 1 & 0 & a \end{bmatrix}$$

   For which real numbers a is the matrix A invertible?

2. Find A^{-1} when $a = 1$.

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Solution

$$\det A = \begin{vmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 1 & 0 & a \end{vmatrix} = a \begin{vmatrix} a & 1 \\ 0 & a \end{vmatrix} - \begin{vmatrix} 0 & 1 \\ 1 & a \end{vmatrix} = a^3 + 1$$

So A is invertible if and only if $a \neq -1$.
If $a = 1$, then

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) = \frac{1}{2} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}.$$

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Plan for next week

Wednesday we shall introduce and study

- abstract *vector spaces* and *subspaces*,
- *null spaces*, *column spaces* and general *linear transformations*.

Sections 4.1–4.2 in “Linear Algebras and Its Applications” (pages 189–208).

Thursday we shall introduce and study

- *linear independence* and *bases* in general vector spaces,
- *coordinate systems* in vector spaces relative to bases.

Section 4.3–4.4 in “Linear Algebras and Its Applications” (pages 208–225).

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