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TMA4115 - Calculus 3

Lecture 15, March 6

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Review of last week's lecture

Last week we looked at

- how to *add* and *multiply* matrices,
- *invertible* matrices and their *inverses*,
- *the invertible matrix theorem*.

Today's lecture

Today we shall introduce and study *determinants*.

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The inverse of an invertible 2×2 matrix

Recall the following result from last week:

Theorem 4

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad - bc = 0$, then A is not invertible.

The number $ad - bc$ is called the *determinant* of A.

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Determinant

- The determinant is a value associated with a square matrix.
- A square matrix A is invertible if and only if $\det(A) \neq 0$.
- The determinant can be used to give an explicit formula for the inverse of an invertible matrix.
- $\det(AB) = \det(A)\det(B)$.
- The absolute value of the determinant gives the scale factor by which area or volume is multiplied under the associated linear transformation.

The definition of the determinant

For any square matrix A, let A_{ij} denote the submatrix formed by deleting the ith row and the jth column of A.

Definition

The *determinant* of a 1×1 matrix $A = [a]$ is $\det(A) = a$. For $n \geq 2$, the *determinant* of an $n \times n$ matrix $A = [a_{ij}]$ is

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(A_{1j}).$$

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Example

Let us compute the determinant of $A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$.

$$\begin{aligned} \det(A) &= a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + a_{13}\det(A_{13}) \\ &= 1 \cdot \begin{vmatrix} 4 & -1 \\ -2 & 0 \end{vmatrix} - 5 \cdot \begin{vmatrix} 2 & -1 \\ 0 & 0 \end{vmatrix} + 0 \cdot \begin{vmatrix} 2 & 4 \\ 0 & -2 \end{vmatrix} \\ &= -2. \end{aligned}$$

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Cofactor expansions

When $A = [a_{ij}]$, the (i,j)-cofactor of A is the number $C_{ij} = (-1)^{i+j} \det(A_{ij})$.

Theorem 1

Let $A = [a_{ij}]$ be an $n \times n$ matrix. Then

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

and

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

for any i and any j between 1 and n.

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Proof of Theorem 1

We will prove the theorem by induction over n.
The theorem is obviously true for $n = 1$.
Assume that $k > 1$ and that the theorem is true for $n = k - 1$.
Let A be a $k \times k$ matrix, let h be an integer between 1 and k, and let i be an integer between 2 and k. Then A_{1h} is a $(k-1) \times (k-1)$ matrix, so

$$\det A_{1h} = \sum_{j=1}^{h-1} (-1)^{i+j} a_{ij} \det(A_{1h})_{ij} + \sum_{j=h+1}^{k} (-1)^{i+j} a_{ij} \det(A_{1h})_{ij}$$

by the induction assumption.

Proof of Theorem 1 (cont.)

We furthermore have that if j is an integer between 1 and k different from h, then $(A_{1h})_{ij} = (A_{ij})_{1h}$. Thus

$$\begin{aligned}\sum_{j=1}^{k} a_{ij} C_{ij} &= \sum_{j=1}^{k} (-1)^{i+j} a_{ij} \det A_{ij} \\ &= \sum_{j=1}^{k} (-1)^{i+j} a_{ij} \left(\sum_{h=1}^{j-1} (-1)^{1+h} a_{1h} \det(A_{ij})_{1h} \right. \\ &\qquad \left. + \sum_{h=j+1}^{k} (-1)^{1+h} a_{1h} \det(A_{ij})_{1h} \right)\end{aligned}$$

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Proof of Theorem 1 (cont.)

$$= \sum_{j=1}^{k} (-1)^{i+j} a_{ij} \left(\sum_{h=1}^{j-1} (-1)^{1+h} a_{1h} \det(A_{1h})_{ij} + \sum_{h=j+1}^{k} (-1)^{1+h} a_{1h} \det(A_{1h})_{ij} \right)$$

$$= \sum_{h=1}^{k} (-1)^{1+h} a_{1h} \left(\sum_{j=1}^{h-1} (-1)^{i+j} a_{ij} \det(A_{1h})_{ij} + \sum_{j=h+1}^{k} (-1)^{i+j} a_{+j} \det(A_{1h})_{ij} \right)$$

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Proof of Theorem 1 (cont.)

$$= \sum_{j=1}^{k} (-1)^{i+j} a_{ij} \det A_{1h}$$
$$= \det A.$$

Similarly, if j is an integer between 1 and k, then

$$\sum_{i=1}^{k} a_{ij} C_{ij} = \sum_{i=1}^{k} (-1)^{i+j} a_{ij} \det A_{ij}$$
$$= \sum_{i=1}^{j-1} (-1)^{i+j} a_{ij} \det A_{ij} + a_{jj} \det A_{jj} +$$
$$\sum_{i=j+1}^{k} (-1)^{i+j} a_{ij} \det A_{ij}$$

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Proof of Theorem 1 (cont.)

$$= \sum_{i=1}^{j-1} (-1)^{i+j} a_{ij} \left(\sum_{h=1}^{j-1} (-1)^{j+h} a_{jh} \det(A_{ij})_{jh} \right.$$

$$\left. + \sum_{h=j+1}^{k} (-1)^{j+h} a_{jh} \det(A_{ij})_{jh} \right)$$

$$+ a_{jj} \det A_{jj}$$

$$+ \sum_{i=j+1}^{k} (-1)^{i+j} a_{ij} \left(\sum_{h=1}^{j-1} (-1)^{j+h} a_{jh} \det(A_{ij})_{jh} \right.$$

$$\left. + \sum_{h=j+1}^{k} (-1)^{j+h} a_{jh} \det(A_{ij})_{jh} \right)$$

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Proof of Theorem 1 (cont.)

$$= \sum_{i=1}^{j-1} (-1)^{i+j} a_{ij} \left(\sum_{h=1}^{j-1} (-1)^{j+h} a_{jh} \det(A_{jh})_{ij} + \sum_{h=j+1}^{k} (-1)^{j+h} a_{jh} \det(A_{jh})_{ij} \right)$$

$$+ a_{jj} \det A_{jj}$$

$$+ \sum_{i=j+1}^{k} (-1)^{i+j} a_{ij} \left(\sum_{h=1}^{j-1} (-1)^{j+h} a_{jh} \det(A_{jh})_{ij} + \sum_{h=j+1}^{k} (-1)^{j+h} a_{jh} \det(A_{jh})_{ij} \right)$$

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Proof of Theorem 1 (cont.)

$$
\begin{aligned}
&= \sum_{h=1}^{j-1} (-1)^{j+h} a_{jh} \Bigg(\sum_{i=1}^{j-1} (-1)^{i+j} a_{ij} \det(A_{jh})_{ij} \\
&\qquad\qquad + \sum_{i=j+1}^{k} (-1)^{i+j} a_{ij} \det(A_{jh})_{ij} \Bigg) \\
&\quad + a_{jj} \det A_{jj} \\
&\quad + \sum_{h=j+1}^{k} (-1)^{j+h} a_{hj} \Bigg(\sum_{i=1}^{j-1} (-1)^{i+j} a_{ij} \det(A_{jh})_{ij} \\
&\qquad\qquad + \sum_{i=j+1}^{k} (-1)^{i+j} a_{ij} \det(A_{jh})_{ij} \Bigg)
\end{aligned}
$$

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Proof of Theorem 1 (cont.)

$$= \sum_{h=1}^{j-1} (-1)^{j+h} a_{jh} \det A_{jh} + a_{jj} \det A_{jj} + \sum_{h=j+1}^{k} (-1)^{j+h} a_{hj} \det A_{jh}$$

$$= \sum_{h=1}^{k} (-1)^{j+h} a_{jh} \det A_{jh} = \det A.$$

Thus it follows by induction that the theorem is true for all n.

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Example

Let us compute the determinant of $A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$ by using a cofactor expansion across the third column.

$$\begin{aligned} \det(A) &= (-1)^{2+3}(-1) \begin{vmatrix} 1 & 5 \\ 0 & -2 \end{vmatrix} \\ &= \begin{vmatrix} 1 & 5 \\ 0 & -2 \end{vmatrix} \\ &= -2 \end{aligned}$$

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Example

Let us compute the determinant of $A = \begin{bmatrix} 1 & 5 & 3 & -2 \\ 0 & 2 & 5 & -1 \\ 0 & 0 & -4 & 5 \\ 0 & 0 & 0 & -3 \end{bmatrix}$.

$$\det(A) = 1 \cdot \begin{vmatrix} 2 & 5 & -1 \\ 0 & -4 & 5 \\ 0 & 0 & -3 \end{vmatrix} = 1 \cdot 2 \cdot \begin{vmatrix} -4 & 5 \\ 0 & -3 \end{vmatrix}$$
$$= 1 \cdot 2 \cdot (-4) \det([-3]) = 1 \cdot 2 \cdot (-4) \cdot (-3)$$
$$= 24.$$

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The determinant of a triangular matrix

A *triangular* matrix is a square matrix $A = [a_{ij}]$ for which $a_{ij} = 0$ when $i > j$.

Theorem 2

If A is a triangular matrix, then $\det(A)$ is the product of the entries on the main diagonal of A.

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Proof of Theorem 2

We will prove the theorem by induction over the number n of rows (and columns) of A.
If $n = 1$, then $\det A = a_{11}$, so the theorem is true in this case.
Suppose $n > 1$ and that the theorem is true for $(n-1) \times (n-1)$ matrices. Then

$$\det A = \begin{vmatrix} a_{11} & 0 \\ 0 & A_{11} \end{vmatrix} = a_{11} \det A_{11} = a_{11} a_{22} \dots a_{nn}.$$

So it follows by induction that the theorem is true for all matrices A.

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Properties of determinants

Theorem 3

Let A be a square matrix.

1. If a multiple of one row of A is added to another row to produce a matrix B, then $\det(B) = \det(A)$.
2. If two rows of A are interchanged to produce B, then $\det(B) = -\det(A)$.
3. If one row of A is multiplied by k to produce B, then $\det(B) = k\det(A)$.

Proof of Theorem 3

Let A be an $n \times n$ matrix and let E be an elementary $n \times n$ matrix. We will show that

$$\det E = \begin{cases} 1 & \text{if } E \text{ is a row replacement matrix,} \\ -1 & \text{if } E \text{ is a row interchange matrix,} \\ k & \text{if } E \text{ is a scale a row by } k \text{ matrix,} \end{cases}$$

and that $\det(EA) = \det E \det A$.
We will prove this by induction over n.
If $n = 1$, then $E = [k]$ for some number k, and then $\det(E) = k$ and $\det(EA) = \det(kA) = k \det A$. So the statement is true for $n = 1$.

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Proof of Theorem 3 (cont.)

Suppose $n = 2$. If $E = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\det E = 1$ and

$$\det(EA) = \det \begin{bmatrix} a + kc & b + kd \\ c & d \end{bmatrix} = (a + kc)d - c(b + kd)$$

$$= ad + kcd - cd - ckd = ad - cd = \det(A).$$

One can in a similarly way prove that if $E = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$, then $\det E = 1$ and $\det(EA) = \det A$, that if $E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then $\det E = -1$ and $\det(EA) = -\det A$,

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Proof of Theorem 3 (cont.)

and that if $E = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$ or $E = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$, then $\det E = k$ and $\det(EA) = k \det A$, so the statement is true for $n = 2$.
Suppose that $k > 2$ and that the statement holds for $n = k - 1$. Let E be an elementary $k \times k$ matrix. Choose i such that the ith row of E is equal to the ith row of I_k. Then E_{ii} is an elementary $(k-1) \times (k-1)$ matrix of the same kind as E, and $\det E = (-1)^{i+i} \det E_{ii} = \det E_{ii}$.

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Proof of Theorem 3 (cont.)

If A is a $k \times k$, then

$$\det(EA) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(EA)_{ij} = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(E_{ii} A_{ij})$$

$$= \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det E_{ii} \det A_{ij})$$

$$= \det E \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij} = \det E \det A.$$

It follows by induction over n that the statement, and thus the theorem, holds for all n.

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Properties of determinants

- Suppose an $n \times n$ matrix A has been reduced to an echelon form U by row replacements and row interchanges.
- If there are r interchanges, then $\det(A) = (-1)^r \det(U)$.
- Since U is in echelon form, it is triangular, so $\det(U)$ is the product of the diagonal entries $u_{11}, u_{22}, \ldots, u_{nn}$.
- If A is invertible, the entries $u_{11}, u_{22}, \ldots, u_{nn}$ are all pivots. Otherwise, at least one u_{ii} is zero.
- Thus,

$$\det(A) = \begin{cases} (-1)^r u_{11} u_{22} \ldots u_{nn} & \text{when } A \text{ is invertible} \\ 0 & \text{when } A \text{ is not invertible} \end{cases}$$

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Properties of determinants (cont.)

Thus we have proved:

Theorem 4

A square matrix A is invertible if and only if $\det(A) \neq 0$.

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Column operations

Theorem 5

If A is a square matrix, then $\det(A^T) = \det(A)$.

Proof of Theorem 5

We will prove the theorem by induction over n where n is the number of rows of A.

If $n = 1$, then $A^T = A$ from which it follows that $\det(A) = \det(A^T)$.

Let k be a positive integer and assume that the theorem is true for all $k \times k$ matrices. Let $n = k + 1$ and let A be an $n \times n$ matrix.

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Proof of Theorem 5 (cont.)

Then

$$\det(A) = \sum_{i=1}^{n} (-1)^{1+i} a_{i1} \det(A_{i1}) = \sum_{i=1}^{n} (-1)^{1+i} a_{i1} \det((A_{i1})^T)$$

$$= \sum_{i=1}^{n} (-1)^{1+i} a_{i1} \det((A^T)_{1i})$$

$$= \sum_{i=1}^{n} (-1)^{1+i} (a^T)_{1i} \det((A^T)_{1i}) = \det(A^T).$$

It follows by induction that $\det(A) = \det(A^T)$ for all square matrices.

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Example

Let us compute the determinant of $A = \begin{bmatrix} 10 & 1 & -7 \\ 3 & 2 & -3 \\ -5 & 0 & 5 \end{bmatrix}$.

It follows from Theorem 5 and Theorem 3 that if we add the third column to the first row, then that does not change the determinant. So

$$\det(A) = \begin{vmatrix} 10 & 1 & -7 \\ 3 & 2 & -3 \\ -5 & 0 & 5 \end{vmatrix} = \begin{vmatrix} 3 & 1 & -7 \\ 0 & 2 & -3 \\ 0 & 0 & 5 \end{vmatrix} = 2 \cdot 3 \cdot 5 = 30.$$

Multiplicative property

Theorem 6

If A and B are $n \times n$ matrices, then $\det(AB) = \det(A)\det(B)$.

It follows from the theorem that if A and B are $n \times n$ matrices, then $\det(AB) = det(A)\det(B) = \det(B)\det(A) = \det(BA)$, even if $AB \neq BA$.

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Proof of Theorem 6

If A is not invertible, then neither is AB, so $\det(A)\det(B) = 0 = \det(AB)$ in that case.
If A is invertible, then A is row equivalent to I_n, so there are elementary matrices $E_1, E_2, \ldots, E_{p-1}, E_p$ such that $A = E_p E_{p-1} \ldots E_2 E_1 I_n = E_p E_{p-1} \ldots E_2 E_1$, and then

$$\begin{aligned}
\det(AB) &= \det(E_p E_{p-1} \ldots E_2 E_1 B) \\
&= \det(E_p)\det(E_{p-1} \ldots E_2 E_1 B) \\
&= \cdots = \det(E_p)\det(E_{p-1}) \ldots \det(E_2)\det(E_1)\det(B) \\
&= \det(E_p E_{p-1} \ldots E_2 E_1)\det(B) \\
&= \det(A)\det(B).
\end{aligned}$$

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Warnings

Let A and B be $n \times n$ matrices and let k be a scalar.

- In general, $\det(A + B) \neq \det(A) + \det(B)$.
- In general, $\det(kA) \neq k \det(A)$.

In fact, $\det(kA) = k^n \det(A)$.

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Problem 4 from June 2005

Find the determinant of the matrix $A = \begin{bmatrix} -2 & 0 & 0 & 8 \\ 1 & -2 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -2 \end{bmatrix}$.

Solution

$$\det A = \begin{vmatrix} -2 & 0 & 0 & 8 \\ 1 & -2 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -2 \end{vmatrix} = \begin{vmatrix} -2 & 2 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -2 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -2 \end{vmatrix} = - \begin{vmatrix} -2 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 1 & -2 \end{vmatrix}$$

$$= 2 \begin{vmatrix} -2 & 0 \\ 1 & -2 \end{vmatrix} = 8.$$

Problem 6 from August 2010

For which values of the parameter a are the vectors $\mathbf{v}_1 = (1, -3, a)$, $\mathbf{v}_2 = (0, 1, a)$ and $\mathbf{v}_3 = (a, 2, 0)$ linearly dependent?

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Solution

$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent if and only if the matrix
$A = \begin{bmatrix} 1 & 0 & a \\ -3 & 1 & 2 \\ a & a & 0 \end{bmatrix}$ is not invertible.

$$\det A = \begin{vmatrix} 1 & 0 & a \\ -3 & 1 & 2 \\ a & a & 0 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ a & 0 \end{vmatrix} + a \begin{vmatrix} -3 & 1 \\ a & a \end{vmatrix} = -2a - 4a^2$$
$$= -2a(1 + 2a).$$

So $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent if and only if $a = 0$ or $a = -1/2$.

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Problem 6 from June 2012

Let A be a 4×4 matrix. Let $B = \begin{bmatrix} 2 & 1 & 4 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Assume that $\det(AB) = 4$. What is $\det(A)$?

Show that the equation $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ has only the solution $x_1 = x_2 = x_3 = x_4 = 0$.

Solution

$$\det B = \begin{vmatrix} 2 & 1 & 4 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 4 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 4 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}$$

$$= -3 + 1 = -2.$$

It follows that $\det A = \frac{\det(AB)}{\det B} = \frac{4}{-2} = -2.$

Solution (cont.)

Since $\det A \neq 0$, A is invertible, so the equation

$$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ has only the solution } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = A^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Tomorrow's lecture

Tomorrow we shall

- look at *Cramer's rule*,
- give a formula for the inverse of an invertible matrix,
- look at the relationship between areas, volumes and determinants.

Section 3.3 in “Linear Algebras and Its Applications” (pages 177–187).