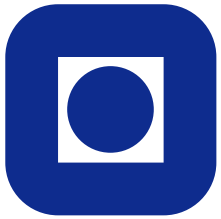

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TMA4115 - Calculus 3

Lecture 12, Feb 21

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Yesterday's lecture

Yesterday we introduced and studied

- *linear transformations*,
- the *standard matrix* of a linear transformation,
- *onto* linear transformations,
- *one-to-one* linear transformations.

Today's lecture

Today we shall

- look at *applications* of linear models,
- look at *Maple* and *WolframAlpha*.

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Linear transformations

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is *linear* if:

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^n$;
2. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all scalars c and all $\mathbf{u}$ in $\mathbb{R}^n$.

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The matrix of a linear transformation

Theorem 10

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$. In fact, if we for each $j = 1, \ldots n$ let $\mathbf{e}_j$ be the jth column of the identity matrix I_n, then A is the $m \times n$ matrix $[T(\mathbf{e}_1) \ \ldots \ T(\mathbf{e}_n)]$ whose jth column is the vector $T(\mathbf{e}_j)$.

The matrix A is called the *standard matrix* of T.

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One-to-one transformations

Definition of one-to-one transformations

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be *one-to-one* (or *injective*) if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of at most one $\mathbf{x}$ in $\mathbb{R}^n$.

Notice that each $\mathbf{b}$ in $\mathbb{R}^m$ does not have to be in the image of T in order for T to be one-to-one.

Theorem 11

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if and only the equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

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Onto transformations

Definition of onto transformations

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be *onto* (or *surjective*) if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of at least one $\mathbf{x}$ in $\mathbb{R}^n$.

Notice that a transformation T is onto if and only if the image of T is all of $\mathbb{R}^m$.

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Onto and one-to-one linear transformations

Theorem 12

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let A be the standard matrix for T. Then:

1. T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$.
2. T is one-to-one if and only if the columns of A are linearly independent.

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Example

Define $T : \mathbb{R}^3 \to \mathbb{R}^2$ by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ for $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$.

It is easy to check that T is linear and that the standard matrix of T is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

Since the columns of the standard matrix of T are linearly dependent and span $\mathbb{R}^2$, it follows that T is onto, but not one-to-one.

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Example

In a certain region, 5% of a city's population moves to the surrounding suburbs each year, and 3% of the suburban population moves into the city. In 2000, there were 600,000 residents in the city and 400,000 in the suburbs. Let us fin the population in the city in the suburbs for the years 2001 and 2002.

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Solution

For $i = 0, 1, 2$, let r_i be the population in the city in year $2000 + i$, and let s_i be the population in the suburbs in year $2000 + i$. Then

$$r_{i+1} = 0.95r_i + 0.03s_i$$
$$s_{i+1} = 0.005r_i + 0.97s_i$$

So if we let $\mathbf{x}_i = \begin{bmatrix} r_i \\ s_i \end{bmatrix}$, then $\mathbf{x}_{i+1} = \begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix} \mathbf{x}_i$.

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Solution (cont.)

$$\mathbf{x}_0 = \begin{bmatrix} 600,000 \\ 400,000 \end{bmatrix} \text{ so}$$

$$\mathbf{x}_1 = \begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix} \begin{bmatrix} 600,000 \\ 400,000 \end{bmatrix} = \begin{bmatrix} 582,000 \\ 418,000 \end{bmatrix} \text{ and}$$

$$\mathbf{x}_2 = \begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix} \begin{bmatrix} 582,000 \\ 418,000 \end{bmatrix} = \begin{bmatrix} 565,440 \\ 434,560 \end{bmatrix}.$$

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Example

The amounts of protein, carbohydrate and fat in nonfat milk, soy flour and whey are given by the following table.

Nutrient	Nonfat milk	Soy flour	Whey
Protein	36	51	13
Carbohydrate	52	34	74
Fat	0	7	1.1

Table: Amounts (g) supplied per 100 g of ingredient

Let us find the amounts of nonfat milk, soy flour and whey that would give a product containing 33 g protein, 45 g carbohydrate and 3 g fat.

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Solution

Let x_1, x_2 and x_3 denote the number of units (100 g) of nonfat milk, soy flour, and whey, respectively, in the product. Then we have that

$$\begin{aligned} 36x_1 + 51x_2 + 13x_3 &= 33 \\ 52x_1 + 34x_2 + 74x_3 &= 45 \\ 7x_2 + 1.1x_3 &= 3 \end{aligned}$$

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Solution (cont.)

We write down the augmented matrix of the system and reduce it to reduced echelon form.

$$\begin{bmatrix} 36 & 51 & 13 & 33 \\ 52 & 34 & 74 & 45 \\ 0 & 7 & 1.1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0.277 \\ 0 & 1 & 0 & 0.392 \\ 0 & 0 & 1 & 0.233 \end{bmatrix}$$

We see that $x_1 = 0.277$, $x_2 = 0.392$ and $x_3 = 0.233$, so the product must consists of 27.7 g nonfat milk, 39.2 g soy flour, and 23.3 g whey.

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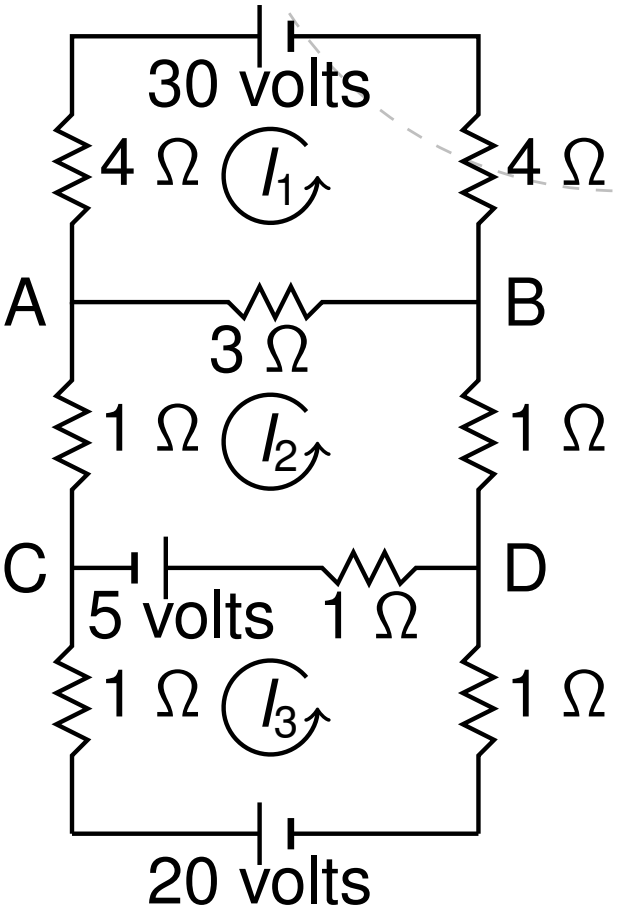
Example

Kirchhoff's voltage law says that the directed sum of the electrical potential differences (voltage) around any closed network is zero.
Ohm's law says that the electrical potential difference across a resistor is the product of the resistance and the current flow.
Let us use these laws to determine the current flows I_1, I_2, and I_3 in the following network.



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Solution

We have that

$$\begin{aligned} 4I_1 + 3(I_1 - I_2) + 4I_1 - 30 &= 0 \\ I_2 + (I_2 - I_3) + I_2 + 3(I_2 - I_1) - 5 &= 0 \\ I_3 + 20 + I_3 + (I_3 - I_2) + 5 &= 0 \end{aligned}$$

which is equivalent to

$$\begin{aligned} 11I_1 - 3I_2 &= 30 \\ -3I_1 + 6I_2 - I_3 &= 5 \\ -I_2 + 3I_3 &= -25 \end{aligned}$$

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Solution (cont.)

We write down the augmented matrix of the system and reduce it to its reduced echelon form.

$$\begin{bmatrix} 11 & -3 & 0 & 30 \\ -3 & 6 & -1 & 5 \\ 0 & -1 & 3 & -25 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -8 \end{bmatrix}$$

It follows that $I_1 = 3$ amps, $I_2 = 1$ amps, and that $I_3 = -8$ amps. That I_3 is negative means that current in loop 3 flows in the direction opposite to the one indicated in the figure.

Mathematical software

Mathematical software is software used to model, analyze or calculate numeric, symbolic or geometric data.
There are two main types of mathematical software:

1. Numerical analysis software
2. Computer algebra systems

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Numerical analysis software

Numerical analysis software is primarily intended for numerical computing.
Generally, numerical analysis software can be used to

- algorithmic numerical calculations
- plotting of functions and data
- finding numerical solutions of differential equations.

Numerical analysis software is primarily used in applied mathematics. *MATLAB* is the main numerical analysis software.

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Computer algebra systems

A *computer algebra system* (CAS) is a software program that facilitates symbolic mathematics. The core functionality of a CAS is manipulation of mathematical expressions in symbolic form.

Computer algebra systems can be used to

- manipulate and simplify mathematical expressions,
- solve equations,
- differentiation and integration,
- solve differential equations.

The main computer algebra systems are *Maple* and *Mathematica*.

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WolframAlpha

WolframAlpha is an online service that answers factual queries directly by computing the answer from structured data, rather than providing a list of documents or web pages that might contain the answer as a search engine might. It is based on the computer algebra system *Mathematica*.

Plan for next week

Wednesday we shall

- see how we can *add* and *multiply* matrices,
- define and study *invertible* matrices.

Sections 2.1–2.2 in “Linear Algebras and Its Applications” (pages 91–111).

Thursday we shall

- further study *invertible* matrices,
- look at *the invertible matrix theorem*.

Section 2.3 in “Linear Algebras and Its Applications” (pages 111–116).

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