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TMA4115 - Calculus 3

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Review of last week's lecture

Last week we

- introduced and solved *homogeneous* and *nonhomegeneous* matrix equations,
- learned how to write solution sets in *parametric vector form*,
- looked at applications of linear systems,
- introduced and studied *linear dependence* and *linear independence* of vectors.

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Today's lecture

We shall introduce and study

- *linear transformations*,
- the *standard matrix* of a linear transformation,
- *onto* linear transformations,
- *one-to-one* linear transformations.

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Transformations

- A *transformation* (or *function* or *mapping*) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ a vector $T(\mathbf{x})$ in $\mathbb{R}^m$.
- The set $\mathbb{R}^n$ is called domain of T, and $\mathbb{R}^m$ is called the codomain of T.
- The notation $T : \mathbb{R}^n \to \mathbb{R}^m$ indicates that the domain of T is $\mathbb{R}^n$ and the codomain is $\mathbb{R}^m$.
- For $\mathbf{x}$ in $\mathbb{R}^n$, the vector $T(\mathbf{x})$ in $\mathbb{R}^m$ is called the image of $\mathbf{x}$ (under the action of T).
- The set of all images $T(\mathbf{x})$ is called the range (or image) of T.

Example

For each $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$, let

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} \sqrt{x_1^2 + x_2^2 + x_3^2} \\ \cos(x_1 x_2 x_3 \pi) \end{bmatrix}$$

Then T is a transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$.

The image of $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is $\begin{bmatrix} \sqrt{1^2 + 1^2 + 1^2} \\ \cos(\pi) \end{bmatrix} = \begin{bmatrix} \sqrt{3} \\ -1 \end{bmatrix}$.

Example (cont.)

The range of T is $\left\{ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} : y \geq 0,\ y_2 \in [-1, 1] \right\}$ (this is not completely obvious).

If $f_1(x_1, x_2, x_3) = \sqrt{x_1^2 + x_2^2 + x_3^2}$ and $f_2(x_1, x_2, x_3) = \cos(x_1 x_2 x_3 \pi)$, then f_1 and f_2 are real-valued functions on $\mathbb{R}^3$ and

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} f_1(x_1, x_2, x_3) \\ f_2(x_1, x_2, x_3) \end{bmatrix}$$

for all $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$.

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Transformations and real-valued functions of several variables

If $f_1, f_2, \ldots, f_m$ are real-valued functions on $\mathbb{R}^n$, then

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) = \begin{bmatrix} f_1(x_1, x_2, \ldots, x_n) \\ f_2(x_1, x_2, \ldots, x_n) \\ \vdots \\ f_m(x_1, x_2, \ldots, x_n) \end{bmatrix}$$

defines a transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

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Transformations and real-valued functions of several variables

Conversely, if T is a transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$, then there exist real-valued functions $f_1, f_2, \ldots, f_m$ on $\mathbb{R}^n$ such that

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) = \begin{bmatrix} f_1(x_1, x_2, \ldots, x_n) \\ f_2(x_1, x_2, \ldots, x_n) \\ \vdots \\ f_m(x_1, x_2, \ldots, x_n) \end{bmatrix}$$

for all $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$.

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Example

For each $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$, let

$$S\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - x_2 \\ x_1 \\ 3x_2 \end{bmatrix}$$

Then S is a transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$.
Notice that

$$S\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = x_1 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Example (cont.)

It follows that the range of S is $\text{Span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} \right\}$.

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Matrix transformations

If A is an $m \times n$ matrix and we for every $\mathbf{x}$ in $\mathbb{R}^n$ let $T(\mathbf{x}) = A\mathbf{x}$, then T is a transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$. The range of A is $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ where $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ are the columns of A.

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Linear transformations

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is *linear* if:

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^n$;
2. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all scalars c and all $\mathbf{u}$ in $\mathbb{R}^n$.

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Matrix transformations are linear

Every matrix transformation is linear because if A is an $m \times n$ matrix, then $A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v}$ for all $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^n$ and $A(c\mathbf{u}) = cA\mathbf{u}$ for all scalars c and all $\mathbf{u}$ in $\mathbb{R}^n$.

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Properties of linear transformations

If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then:

1. $T(\mathbf{0}) = \mathbf{0}$ because $T(\mathbf{0}) = T(0\mathbf{u}) = 0T(\mathbf{u}) = \mathbf{0}$ for any vector $\mathbf{u}$ in $\mathbb{R}^n$.
2. $T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$ for all scalars c and d and all vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, because $T(c\mathbf{u} + d\mathbf{v}) = T(c\mathbf{u}) + T(d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$.

Example

A company manufactures two products. For \$1.00 worth of product B, the company spends \$0.45 on material, \$0.25 on labor, and \$0.15 on overhead. For \$1.00 worth of product C, the company spends \$0.40 on material, \$0.30 on labor, and \$0.15 on overhead.

Let x_1 denote the value of product B and x_2 the value of product C that the company manufactures.

Let y_1, y_2 and y_3 denote the costs of material, labor and overhead the company spends on this.

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Example (cont.)

Then

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0.45x_1 + 0.40x_2 \\ 0.25x_1 + 0.30x_2 \\ 0.15x_1 + 0.15x_2 \end{bmatrix}$$

So if we define $T : \mathbb{R}^2 \to \mathbb{R}^3$ by

$$T(\mathbf{x}) = \begin{bmatrix} 0.45 & 0.40 \\ 0.25 & 0.30 \\ 0.15 & 0.15 \end{bmatrix} \mathbf{x}$$

then $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = T(\mathbf{x})$.

Example

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\3\\-1\end{bmatrix} \text{ and } T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}-3\\5\\7\end{bmatrix}.$$

1. Let $\mathbf{u} = \begin{bmatrix}2\\-1\end{bmatrix}$. Let us find the image $T(\mathbf{u})$ of $\mathbf{u}$ under T.

$$T(\mathbf{u}) = T\left(2\begin{bmatrix}1\\0\end{bmatrix} - \begin{bmatrix}0\\1\end{bmatrix}\right) = 2T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) - T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$$

$$= 2\begin{bmatrix}1\\3\\-1\end{bmatrix} - \begin{bmatrix}-3\\5\\7\end{bmatrix} = \begin{bmatrix}5\\1\\-9\end{bmatrix}$$

Example (cont.)

2. Let us find the image under T of $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

$$T(\mathbf{x}) = T\left(x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = x_1 T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + x_2 T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$= x_1 \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} x_1 - 3x_2 \\ 3x_1 + 5x_2 \\ -x_1 + 7x_2 \end{bmatrix}$$

It follows that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^2$ where

$$A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}.$$

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Example (cont.)

3. Is there more than one $\mathbf{x}$ whose image under T is $\begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}$?

   The question is equivalent to the question: Does the equation $A\mathbf{x} = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}$ have more than one solution?

   To answer that question we reduce the augmented matrix of the equation to an echelon form.

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Example (cont.)

$$\begin{bmatrix} 1 & -3 & 5 \\ 3 & 5 & 1 \\ -1 & 7 & -9 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 5 \\ 0 & 14 & -14 \\ 0 & 4 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

We see that there is no free variable, so there is only one $\mathbf{x}$ such that $T(\mathbf{x}) = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}$.

Example (cont.)

4. Let $\mathbf{c} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$. Let us determine if $\mathbf{c}$ is in the range of T.

   $\mathbf{c}$ is in the range of T if and only if the equation $A\mathbf{x} = \mathbf{c}$ is consistent.
   In order to determine whether the equation $A\mathbf{x} = \mathbf{c}$ is consistent or not, we reduce the augmented matrix $[A\ \mathbf{c}]$ to an echelon form.

Example (cont.)

$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 3 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 2 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -5 \end{bmatrix}$$

We see that the system is inconsistent, so $\mathbf{c}$ is not in the range of T.

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The matrix of a linear transformation

Theorem 10

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$. In fact, if we for each $j = 1, \ldots n$ let $\mathbf{e}_j$ be the jth column of the identity matrix I_n, then A is the $m \times n$ matrix $[T(\mathbf{e}_1) \ \ldots \ T(\mathbf{e}_n)]$ whose jth column is the vector $T(\mathbf{e}_j)$.

The matrix A is called the *standard matrix* of T.

Proof

Let $A = [T(\mathbf{e}_1) \ \ldots \ T(\mathbf{e}_n)]$. We must show that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$.

Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ be a vector in $\mathbb{R}^n$.

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Proof (cont.)

Then

$$T(\mathbf{x}) = T\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) = T\left(x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + x_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right)$$

$$= T(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n)$$

$$= x_1 T(\mathbf{e}_1) + x_2 T(\mathbf{e}_2) + \cdots + x_n T(\mathbf{e}_n)$$

$$= A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = A\mathbf{x}.$$

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Example

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that $T(x_1, x_2, x_3) = (x_1 - 3x_2 - x_3, x_2 + 4x_3, 2x_1 + 9x_2 + 5x_3)$. Let us find the standard matrix of T.

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Solution

$$T(\mathbf{e}_1) = T(1,0,0) = (1,0,2) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix},$$

$$T(\mathbf{e}_2) = T(0,1,0) = (-3,1,9) = \begin{bmatrix} -3 \\ 1 \\ 9 \end{bmatrix},$$

$$T(\mathbf{e}_3) = T(0,0,1) = (-1,4,5) = \begin{bmatrix} -1 \\ 4 \\ 5 \end{bmatrix},$$ so the standard

matrix of T is $\begin{bmatrix} 1 & -3 & -1 \\ 0 & 1 & 4 \\ 2 & 9 & 5 \end{bmatrix}$.

Example

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the function given by $T(\mathbf{x}) = \mathbf{y}$ where $\mathbf{y}$ is the vector we obtain from rotating $\mathbf{x}$ by an angle of θ around zero.
Let us show that T is a linear transformation and find its standard matrix.

Solution

Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be a vector in $\mathbb{R}^2$, and let $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = T(\mathbf{x})$.
Let $r = |x_1 + ix_2| = \sqrt{x_1^2 + x_2^2}$ and $\phi = \operatorname{Arg}(x_1 + ix_2)$. Then $x_1 = r\cos\phi$ and $x_2 = r\sin\phi$, so $y_1 = r\cos(\phi + \theta) = r\cos\phi\cos\theta - r\sin\phi\sin\theta = x_1\cos\theta - x_2\sin\theta$ and $y_2 = r\sin(\phi + \theta) = r\sin\phi\cos\theta + r\cos\phi\sin\theta = x_2\cos\theta + x_1\sin\theta$.
So if we let $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, then $T(\mathbf{x}) = A\mathbf{x}$ for every $\mathbf{x}$ in $\mathbb{R}^2$.
It follows that T is linear, and that A is the standard matrix of T.

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One-to-one transformations

Definition of one-to-one transformations

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be *one-to-one* (or *injective*) if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of at most one $\mathbf{x}$ in $\mathbb{R}^n$.

Notice that each $\mathbf{b}$ in $\mathbb{R}^m$ does not have to be in the image of T in order for T to be one-to-one.

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One-to-one transformations

Theorem 11

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if and only the equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

Proof

We have that $T(\mathbf{0}) = \mathbf{0}$ since T is linear.
If T is one-to-one, then there is at most one $\mathbf{x}$ such that $T(\mathbf{x}) = \mathbf{0}$. So the equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.
If T is not one-to-one, then there are two different vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ such that $T(\mathbf{u}) = T(\mathbf{v})$. We then have that $T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v}) = \mathbf{0}$, so $\mathbf{x} = \mathbf{u} - \mathbf{v}$ is a nontrivial solution to the equation $T(\mathbf{x}) = \mathbf{0}$.

Example

Let T be the linear transformation whose standard matrix is

$$A = \begin{bmatrix} 1 & -2 & -4 & 7 \\ 0 & 2 & 6 & -3 \\ -1 & 2 & 4 & 5 \end{bmatrix}$$

Let us determine if T is one-to-one.

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Solution

T is one-to-one if and only if the equation $A\mathbf{x} = \mathbf{0}$ only has the trivial solution. So we reduce A to an echelon form and check if A has a pivot position in every column.

$$\begin{bmatrix} 1 & -2 & -4 & 7 \\ 0 & 2 & 6 & -3 \\ -1 & 2 & 4 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & -4 & 7 \\ 0 & 2 & 6 & -3 \\ 0 & 0 & 0 & 12 \end{bmatrix}$$

Since A does not have a pivot position in every column, the equation $A\mathbf{x} = \mathbf{0}$ has a free position and therefore a nontrivial solution, so T is not one-to-one.

How to determine if a linear transformation is one-to-one

The following procedure outlines how to determine if a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one:

1. Find the standard matrix A of T.
2. Reduce A to an echelon form.
3. If A has a pivot position in every column, then the equation $A\mathbf{x} = \mathbf{0}$ has no free variable, so the equation $A\mathbf{x} = \mathbf{0}$ only has the trivial solution, and T is one-to-one.

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Onto transformations

Definition of onto transformations

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be *onto* (or *surjective*) if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of at least one $\mathbf{x}$ in $\mathbb{R}^n$.

Notice that a transformation T is onto if and only if the image of T is all of $\mathbb{R}^m$.

Example

Let T be the linear transformation whose standard matrix is

$$A = \begin{bmatrix} 1 & -2 & -4 & 7 \\ 0 & 2 & 6 & -3 \\ -1 & 2 & 4 & 5 \end{bmatrix}$$

Let us determine if T is onto.

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Solution

T is onto if and only if the equation $A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b}$ in $\mathbb{R}^3$. We know that the equation $A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b}$ in $\mathbb{R}^3$ if and only if A has a pivot position in every row. So we reduce A to an echelon form and check if A has a pivot position in every row.

$$\begin{bmatrix} 1 & -2 & -4 & 7 \\ 0 & 2 & 6 & -3 \\ -1 & 2 & 4 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -4 & 7 \\ 0 & 2 & 6 & -3 \\ 0 & 0 & 0 & 12 \end{bmatrix}$$

We see that A has a pivot position in every row, so T is onto.

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How to determine if a linear transformation is onto

The following procedure outlines how to determine if a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto:

1. Find the standard matrix A of T.
2. Reduce A to an echelon form.
3. If A has a pivot position in every row, then the equation $A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b}$ in $\mathbb{R}^m$ and T is onto.

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Onto and one-to-one linear transformations

Theorem 12

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let A be the standard matrix for T. Then:

1. T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$.
2. T is one-to-one if and only if the columns of A are linearly independent.

Proof

$$\begin{aligned} T \text{ is onto} \quad &\iff \text{the equation } A\mathbf{x} = \mathbf{b} \text{ has a solution} \\ & \text{ for every } b \in \mathbb{R}^m \\ &\iff \text{the columns of } A \text{ span } \mathbb{R}^m. \end{aligned}$$

$$\begin{aligned} T \text{ is one-to-one} \quad &\iff \text{the equation } A\mathbf{x} = \mathbf{0} \text{ has a} \\ & \text{ nontrivial solution} \\ &\iff \text{the columns of } A \text{ are} \\ & \text{ linearly independent.} \end{aligned}$$

Example

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that $T(x_1, x_2) = (x_1 + 2x_2, 2x_1 + x_2, 0)$. Let us determine if T is onto and if it is one-to-one.

Solution

Every linear combination of $\begin{bmatrix}1\\2\\0\end{bmatrix}$ and $\begin{bmatrix}2\\1\\0\end{bmatrix}$ has a zero as its third entry, so the columns of A does not span $\mathbb{R}^3$. It follows that T is not onto.

Since $\begin{bmatrix}1\\2\\0\end{bmatrix}$ is not multiple of $\begin{bmatrix}2\\1\\0\end{bmatrix}$, and $\begin{bmatrix}2\\1\\0\end{bmatrix}$ is not a multiple of $\begin{bmatrix}1\\2\\0\end{bmatrix}$, the columns of A are linearly independent. It follows that T is one-to-one.

Tomorrow's lecture

Tomorrow we shall

- look at *applications* of linear models,
- look at the use of *Maple* and *WolframAlpha*.

Section 1.10 in "Linear Algebras and Its Applications" (pages 80-90).