

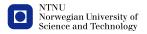
TMA4115 - Calculus 3 Lecture 11, Feb 20

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Review of last week's lecture

Last week we

- introduced and solved *homogeneous* and *nonhomegeneous* matrix equations,
- learned how to write solution sets in *parametric vector* form,
- looked at applications of linear systems,
- introduced and studied *linear dependence* and *linear independence* of vectors.



Today's lecture

We shall introduce and study

- linear transformations,
- the standard matrix of a linear transformation,
- onto linear transformations,
- one-to-one linear transformations.



Transformations

- A transformation (or function or mapping) T from ℝⁿ to ℝ^m is a rule that assigns to each vector **x** in ℝⁿ a vector T(**x**) in ℝ^m.
- The set \mathbb{R}^n is called domain of T, and \mathbb{R}^m is called the codomain of T.
- The notation *T* : ℝⁿ → ℝ^m indicates that the domain of *T* is ℝⁿ and the codomain is ℝ^m.
- For x in ℝⁿ, the vector T(x) in ℝ^m is called the image of x (under the action of T).
- The set of all images $T(\mathbf{x})$ is called the range (or image) of T.



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For each
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$$
, let
 $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} \sqrt{x_1^2 + x_2^2 + x_3^2} \\ \cos(x_1 x_2 x_3 \pi) \end{bmatrix}$
Then *T* is a transformation from \mathbb{R}^3 to \mathbb{R}^2 .

Then *T* is a transformation from \mathbb{R}^3 to \mathbb{R}^2 . The image of $\begin{bmatrix} 1\\1\\1 \end{bmatrix}$ is $\begin{bmatrix} \sqrt{1^2 + 1^2 + 1^2}\\\cos(\pi) \end{bmatrix} = \begin{bmatrix} \sqrt{3}\\-1 \end{bmatrix}$.



for all $\begin{vmatrix} x_1 \\ x_2 \end{vmatrix} \in \mathbb{R}^3$.

The range of *T* is $\left\{ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} : y \ge 0, y_2 \in [-1, 1] \right\}$ (this is not completely obvious). If $f_1(x_1, x_2, x_3) = \sqrt{x_1^2 + x_2^2 + x_3^2}$ and $f_2(x_1, x_2, x_3) = \cos(x_1 x_2 x_3 \pi)$, then f_1 and f_2 are real-valued functions on \mathbb{R}^3 and

$$T\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right) = \begin{bmatrix}f_1(x_1,x_2,x_3)\\f_2(x_1,x_2,x_3)\end{bmatrix}$$



Transformations and real-valued functions of several variables

If f_1, f_2, \ldots, f_m are real-valued functions on \mathbb{R}^n , then

$$T\left(\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}\right) = \begin{bmatrix}f_1(x_1,x_2,\ldots,x_n)\\f_2(x_1,x_2,\ldots,x_n)\\\vdots\\f_m(x_1,x_2,\ldots,x_n)\end{bmatrix}$$

defines a transformation from \mathbb{R}^n to \mathbb{R}^m .



Transformations and real-valued functions of several variables

Conversely, if *T* is a transformation from \mathbb{R}^n to \mathbb{R}^m , then there exist real-valued functions f_1, f_2, \ldots, f_m on \mathbb{R}^n such that

$$T\left(\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}\right) = \begin{bmatrix}f_1(x_1, x_2, \dots, x_n)\\f_2(x_1, x_2, \dots, x_n)\\\vdots\\f_m(x_1, x_2, \dots, x_n)\end{bmatrix}$$



for all

For each
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$$
, let $S\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} 2x_1 - x_2 \\ x_1 \\ 3x_2 \end{bmatrix}$

Then *S* is a transformation from \mathbb{R}^2 to \mathbb{R}^3 . Notice that

$$S\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = x_1\begin{bmatrix}2\\1\\0\end{bmatrix} + x_2\begin{bmatrix}-1\\0\\3\end{bmatrix} = \begin{bmatrix}2 & -1\\1 & 0\\0 & 3\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

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It follows that the range of *S* is Span $\left\{ \begin{bmatrix} 2\\1\\0 \end{bmatrix}, \begin{bmatrix} -1\\0\\3 \end{bmatrix} \right\}$.



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Matrix transformations

If *A* is an $m \times n$ matrix and we for every **x** in \mathbb{R}^n let $T(\mathbf{x}) = A\mathbf{x}$, then *T* is a transformation from \mathbb{R}^n to \mathbb{R}^m . The range of *A* is Span $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$ where $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ are the columns of *A*.



Linear transformations

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is *linear* if:

- $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all \mathbf{u}, \mathbf{v} in \mathbb{R}^n ;
- 2 $T(c\mathbf{u}) = cT(\mathbf{u})$ for all scalars c and all \mathbf{u} in \mathbb{R}^n .



Matrix transformations are linear

Every matrix transformation is linear because if A is an $m \times n$ matrix, then $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$ for all \mathbf{u}, \mathbf{v} in \mathbb{R}^n and $A(c\mathbf{u}) = cA\mathbf{u}$ for all scalars c and all \mathbf{u} in \mathbb{R}^n .



Properties of linear transformations

If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then:

- $T(\mathbf{0}) = \mathbf{0}$ because $T(\mathbf{0}) = T(\mathbf{0}\mathbf{u}) = \mathbf{0}T(\mathbf{u}) = \mathbf{0}$ for any vector \mathbf{u} in \mathbb{R}^n .
- 2 $T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$ for all scalars c and d and all vectors \mathbf{u} and \mathbf{v} in \mathbb{R}^n , because $T(c\mathbf{u} + d\mathbf{v}) = T(c\mathbf{u}) + T(d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$.



A company manufactures two products. For \$1.00 worth of product B, the company spends \$0.45 on material, \$0.25 on labor, and \$0.15 on overhead. For \$1.00 worth of product C, the company spends \$0.40 on material, \$0.30 on labor, and \$0.15 on overhead.

Let x_1 denote the value of product B and x_2 the value of product C that the company manufactures.

Let y_1 , y_2 and y_3 denote the costs of material, labor and overhead the company spends on this.



Then

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0.45x_1 + 0.40x_2 \\ 0.25x_1 + 0.30x_2 \\ 0.15x_1 + 0.15x_2 \end{bmatrix}$$

So if we define $T : \mathbb{R}^2 \to \mathbb{R}^3$ by $T(\mathbf{x}) = \begin{bmatrix} 0.45 & 0.40 \\ 0.25 & 0.30 \\ 0.15 & 0.15 \end{bmatrix} \mathbf{x}$ $\begin{bmatrix} v_1 \end{bmatrix}$

then
$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = T(\mathbf{x}).$$

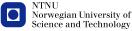


Let
$$T : \mathbb{R}^2 \to \mathbb{R}^3$$
 be a linear transformation such that
 $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\3\\-1\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}-3\\5\\7\end{bmatrix}$.
• Let $\mathbf{u} = \begin{bmatrix}2\\-1\end{bmatrix}$. Let us find the image $T(\mathbf{u})$ of \mathbf{u} under T .
 $T(\mathbf{u}) = T\left(2\begin{bmatrix}1\\0\end{bmatrix} - \begin{bmatrix}0\\1\end{bmatrix}\right) = 2T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) - T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$
 $= 2\begin{bmatrix}1\\3\\-1\end{bmatrix} - \begin{bmatrix}-3\\5\\7\end{bmatrix} = \begin{bmatrix}5\\1\\-9\end{bmatrix}$
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2 Let us find the image under T of $\mathbf{x} = \begin{vmatrix} x_1 \\ x_2 \end{vmatrix}$.

$$T(\mathbf{x}) = T\left(x_1 \begin{bmatrix} 1\\0 \end{bmatrix} + x_2 \begin{bmatrix} 0\\1 \end{bmatrix}\right) = x_1 T\left(\begin{bmatrix} 1\\0 \end{bmatrix}\right) + x_2 T\left(\begin{bmatrix} 0\\1 \end{bmatrix}\right)$$
$$= x_1 \begin{bmatrix} 1\\3\\-1 \end{bmatrix} + x_2 \begin{bmatrix} -3\\5\\7 \end{bmatrix} = \begin{bmatrix} x_1 - 3x_2\\3x_1 + 5x_2\\-x_1 + 7x_2 \end{bmatrix}$$

It follows that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^2$ where $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$.



Solution Is there more than one **x** whose image under *T* is $\begin{bmatrix} 5\\1\\-9 \end{bmatrix}$?

The question is equivalent to the question: Does the equation $A\mathbf{x} = \begin{bmatrix} 5\\1\\-9 \end{bmatrix}$ have more than one solution?

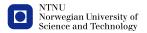
To answer that question we reduce the augmented matrix of the equation to an echelon form.



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$$\begin{bmatrix} 1 & -3 & 5 \\ 3 & 5 & 1 \\ -1 & 7 & -9 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 5 \\ 0 & 14 & -14 \\ 0 & 4 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

We see that there is no free variable, so there is only one **x** such that $T(\mathbf{x}) = \begin{bmatrix} 5\\1\\-9 \end{bmatrix}$.

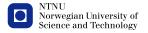


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• Let $\mathbf{c} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$. Let us determine if \mathbf{c} is in the range of \mathcal{T} .

c is in the range of T if and only if the equation $A\mathbf{x} = \mathbf{c}$ is consistent.

In order to determine whether the equation $A\mathbf{x} = \mathbf{c}$ is consistent or not, we reduce the augmented matrix $[A \mathbf{c}]$ to an echelon form.



$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & 8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 3 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$
$$\rightarrow \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 2 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -5 \end{bmatrix}$$

We see that the system is inconsistent, so c is not in the range of T.

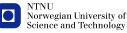


The matrix of a linear transformation

Theorem 10

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for all \mathbf{x} in \mathbb{R}^n . In fact, if we for each j = 1, ..., n let \mathbf{e}_j be the *j*th column of the identity matrix I_n , then A is the $m \times n$ matrix $[T(\mathbf{e}_1) \ldots T(\mathbf{e}_n)]$ whose *j*th column is the vector $T(\mathbf{e}_j)$.

The matrix A is called the standard matrix of T.



Proof

Let $A = [T(\mathbf{e}_1) \dots T(\mathbf{e}_n)]$. We must show that $T(\mathbf{x}) = A\mathbf{x}$ for all \mathbf{x} in \mathbb{R}^n . Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ be a vector in \mathbb{R}^n .



Proof (cont.)

Then

$$T(\mathbf{x}) = T\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \right) = T\left(x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \dots + x_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right)$$
$$= T(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \dots + x_n\mathbf{e}_n)$$
$$= x_1 T(\mathbf{e}_1) + x_2 T(\mathbf{e}_2) + \dots + x_n T(\mathbf{e}_n)$$
$$= A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = A\mathbf{x}.$$

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that $T(x_1, x_2, x_3) = (x_1 - 3x_2 - x_3, x_2 + 4x_3, 2x_1 + 9x_2 + 5x_3)$. Let us find the standard matrix of T.



Solution

$$T(\mathbf{e}_{1}) = T(1,0,0) = (1,0,2) = \begin{bmatrix} 1\\0\\2 \end{bmatrix},$$

$$T(\mathbf{e}_{2}) = T(0,1,0) = (-3,1,9) = \begin{bmatrix} -3\\1\\9\\-1\\4\\5 \end{bmatrix},$$

$$T(\mathbf{e}_{3}) = T(0,0,1) = (-1,4,5) = \begin{bmatrix} -1\\4\\5 \end{bmatrix},$$
 so the standard matrix of T is $\begin{bmatrix} 1 & -3 & -1\\0 & 1 & 4\\2 & 9 & 5 \end{bmatrix}.$

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the function given by $T(\mathbf{x}) = \mathbf{y}$ where \mathbf{y} is the vector we obtain from rotating \mathbf{x} by an angle of θ around zero.

Let us show that T is a linear transformation and find its standard matrix.



Solution

Let
$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
 be a vector in \mathbb{R}^2 , and let $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = T(\mathbf{x})$.
Let $r = |x_1 + ix_2| = \sqrt{x_1^2 + x_2^2}$ and $\phi = \operatorname{Arg}(x_1 + ix_2)$. Then
 $x_1 = r \cos \phi$ and $x_2 = r \sin \phi$, so $y_1 = r \cos(\phi + \theta) =$
 $r \cos \phi \cos \theta - r \sin \phi \sin \theta = x_1 \cos \theta - x_2 \sin \theta$ and $y_2 =$
 $r \sin(\phi + \theta) = r \sin \phi \cos \theta + r \cos \phi \sin \theta = x_2 \cos \theta + x_1 \sin \theta$.
So if we let $A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$, then $T(\mathbf{x}) = A\mathbf{x}$ for every \mathbf{x}
in \mathbb{R}^2 .
It follows that T is linear, and that A is the standard matrix of
 T .



One-to-one transformations

Definition of one-to-one transformations

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be *one-to-one* (or *injective*) if each **b** in \mathbb{R}^m is the image of at most one **x** in \mathbb{R}^n .

Notice that each **b** in \mathbb{R}^m does not have to be in the image of T in order for T to be one-to-one.



One-to-one transformations

Theorem 11

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if and only the equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.



Proof

We have that $T(\mathbf{0}) = \mathbf{0}$ since T is linear.

If *T* is one-to-one, then there is at most one **x** such that $T(\mathbf{x}) = \mathbf{0}$. So the equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

If *T* is not one-to-one, then there are two different vectors \mathbf{u} and \mathbf{v} in \mathbb{R}^n such that $T(\mathbf{u}) = T(\mathbf{v})$. We then have that $T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v}) = \mathbf{0}$, so $\mathbf{x} = \mathbf{u} - \mathbf{v}$ is a nontrivial solution to the equation $T(\mathbf{x}) = \mathbf{0}$.



Let T be the linear transformation whose standard matrix is

$$A = \begin{bmatrix} 1 & -2 & -4 & 7 \\ 0 & 2 & 6 & -3 \\ -1 & 2 & 4 & 5 \end{bmatrix}$$

Let us determine if T is one-to-one.



Solution

T is one-to-one if and only if the equation $A\mathbf{x} = \mathbf{0}$ only has the trivial solution. So we reduce *A* to an echelon form and check if *A* has a pivot position in every column.

$$\begin{bmatrix} 1 & -2 & -4 & 7 \\ 0 & 2 & 6 & -3 \\ -1 & 2 & 4 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & -4 & 7 \\ 0 & 2 & 6 & -3 \\ 0 & 0 & 0 & 12 \end{bmatrix}$$

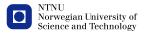
Since *A* does not have a pivot position in every column, the equation $A\mathbf{x} = \mathbf{0}$ has a free position and therefore a nontrivial solution, so *T* is not one-to-one.



How to determine if a linear transformation is one-to-one

The following procedure outlines how to determine if a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one:

- Find the standard matrix A of T.
- Reduce A to an echelon form.
- If A has a pivot position in every column, then the equation Ax = 0 has no free variable, so the equation Ax = 0 only has the trivial solution, and T is one-to-one.



Onto transformations

Definition of onto transformations

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be *onto* (or *surjective*) if each **b** in \mathbb{R}^m is the image of at least one **x** in \mathbb{R}^n .

Notice that a transformation T is onto if and only if the image of T is all of \mathbb{R}^m .



Let T be the linear transformation whose standard matrix is

$$A = \begin{bmatrix} 1 & -2 & -4 & 7 \\ 0 & 2 & 6 & -3 \\ -1 & 2 & 4 & 5 \end{bmatrix}$$

Let us determine if *T* is onto.



Solution

T is onto if and only if the equation $A\mathbf{x} = \mathbf{b}$ is consistent for all \mathbf{b} in \mathbb{R}^3 . We know that the equation $A\mathbf{x} = \mathbf{b}$ is consistent for all \mathbf{b} in \mathbb{R}^3 if and only if *A* has a pivot position in every row. So we reduce *A* to an echelon form and check if *A* has a pivot position in every row.

$$\begin{bmatrix} 1 & -2 & -4 & 7 \\ 0 & 2 & 6 & -3 \\ -1 & 2 & 4 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & -4 & 7 \\ 0 & 2 & 6 & -3 \\ 0 & 0 & 0 & 12 \end{bmatrix}$$

We see that A has a pivot position in every row, so T is onto.



How to determine if a linear transformation is onto

The following procedure outlines how to determine if a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto:

- Find the standard matrix A of T.
- Reduce A to an echelon form.
- If *A* has a pivot position in every row, then the equation $A\mathbf{x} = \mathbf{b}$ is consistent for all **b** in \mathbb{R}^m and *T* is onto.



Onto and one-to-one linear transformations

Theorem 12

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let A be the standard matrix for T. Then:

- T maps \mathbb{R}^n onto \mathbb{R}^m if and only if the columns of A span \mathbb{R}^m .
- T is one-to-one if and only if the columns of A are linearly independent.



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Proof

$\begin{array}{ll} T \text{ is onto} & \Longleftrightarrow \text{ the equation } A\mathbf{x} = \mathbf{b} \text{ has a solution} \\ & \text{ for every } b \in \mathbb{R}^m \\ & \Longleftrightarrow \text{ the columns of } A \text{ span } \mathbb{R}^m. \end{array}$

T is one-to-one \iff the equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution

 \iff the columns of A are

linearly independent.

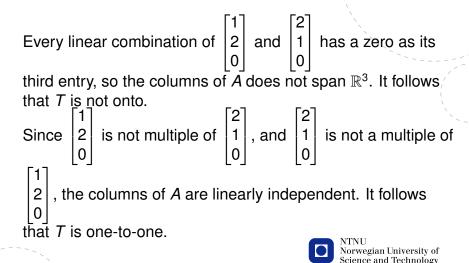


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Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that $T(x_1, x_2) = (x_1 + 2x_2, 2x_1 + x_2, 0)$. Let us determine if T is onto and if it is one-to-one.



Solution



Tomorrow's lecture

Tomorrow we shall

- look at applications of linear models,
- look at the use of Maple and WolframAlpha.

Section 1.10 in "Linear Algebras and Its Applications" (pages 80-90).

