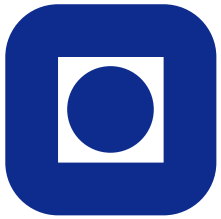


NTNU
Norwegian University of
Science and Technology

TMA4115 - Calculus 3

Lecture 10, Feb 14

Toke Meier Carlsen
Norwegian University of Science and Technology

Spring 2013

Yesterday's lecture

Yesterday we

- introduced and solved *homogeneous* and *nonhomegeneous* matrix equations,
- learned how to write solution sets in *parametric vector form*,
- looked at applications of linear systems.

NTNU
Norwegian University of
Science and Technology

Today's lecture

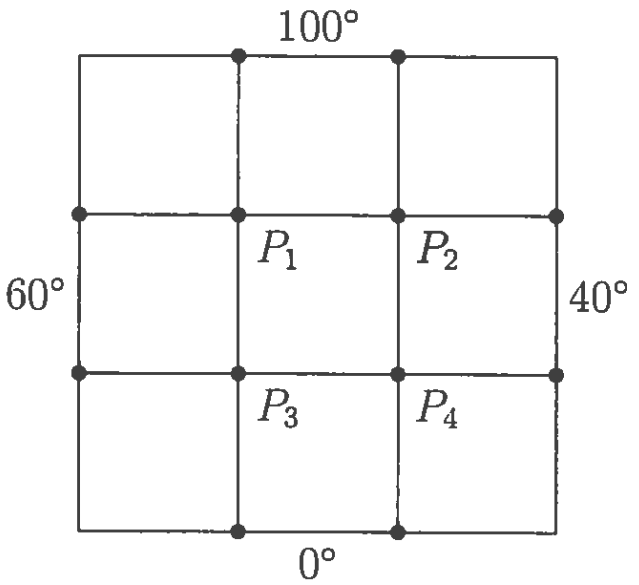
We shall introduce and study *linear dependence* and *linear independence* of vectors.

NTNU
Norwegian University of
Science and Technology

Problem 4 from December 2007

Problem 4 We shall determine the temperatures T_1, T_2, T_3, T_4 at four points P_1, P_2, P_3, P_4 on the square plate, see the figure on the right. Temperatures along the sides of the plate are shown on the figure.

Set up and solve a system of equations for T_1, T_2, T_3, T_4 if we assume that the temperature at each point P_1 to P_4 is the average of the temperatures at the four neighboring points (to the right, above, to the left, and below).



NTNU
Norwegian University of
Science and Technology

Solution

We have that

$$T_1 = \frac{1}{4}(60 + 100 + T_2 + T_3)$$
$$T_2 = \frac{1}{4}(T_1 + 100 + 40 + T_4)$$
$$T_3 = \frac{1}{4}(60 + T_1 + T_4 + 0)$$
$$T_4 = \frac{1}{4}(T_3 + T_2 + 40 + 0)$$

NTNU
Norwegian University of
Science and Technology

Solution (cont.)

which is equivalent to

$$\begin{aligned} 4T_1 - T_2 - T_3 &= 160 \\ -T_1 + 4T_2 - T_4 &= 140 \\ -T_1 + 4T_3 - T_4 &= 60 \\ -T_2 - T_3 + 4T_4 &= 40 \end{aligned}$$

We reduce the augmented matrix of the system to its reduced echelon form.

NTNU
Norwegian University of
Science and Technology

Solution (cont.)

$$\begin{bmatrix} 4 & -1 & -1 & 0 & 160 \\ -1 & 4 & 0 & -1 & 140 \\ -1 & 0 & 4 & -1 & 60 \\ 0 & -1 & -1 & 4 & 40 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -4 & 1 & -60 \\ -1 & 4 & 0 & -1 & 140 \\ 4 & -1 & -1 & 0 & 160 \\ 0 & -1 & -1 & 4 & 40 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & -4 & 1 & -60 \\ 0 & 4 & -4 & 0 & 80 \\ 0 & -1 & 15 & -4 & 400 \\ 0 & -1 & -1 & 4 & 40 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -4 & 1 & -60 \\ 0 & 1 & 1 & -4 & -40 \\ 0 & -1 & 15 & -4 & 400 \\ 0 & 4 & -4 & 0 & 80 \end{bmatrix}$$

NTNU
Norwegian University of
Science and Technology

Solution (cont.)

$$\rightarrow \begin{bmatrix} 1 & 0 & -4 & 1 & -60 \\ 0 & 1 & 1 & -4 & -40 \\ 0 & 0 & 16 & -8 & 360 \\ 0 & 0 & -8 & 16 & 240 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -4 & 1 & -60 \\ 0 & 1 & 1 & -4 & -40 \\ 0 & 0 & 1 & -2 & -30 \\ 0 & 0 & 16 & -8 & 360 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & -4 & 1 & -60 \\ 0 & 1 & 1 & -4 & -40 \\ 0 & 0 & 1 & -2 & -30 \\ 0 & 0 & 0 & 24 & 840 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -4 & 1 & -60 \\ 0 & 1 & 1 & -4 & -40 \\ 0 & 0 & 1 & -2 & -30 \\ 0 & 0 & 0 & 1 & 35 \end{bmatrix}$$

NTNU
Norwegian University of
Science and Technology

Solution (cont.)

$$\rightarrow \begin{bmatrix} 1 & 0 & -4 & 0 & -95 \\ 0 & 1 & 1 & 0 & 100 \\ 0 & 0 & 1 & 0 & 40 \\ 0 & 0 & 0 & 1 & 35 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 65 \\ 0 & 1 & 0 & 0 & 60 \\ 0 & 0 & 1 & 0 & 40 \\ 0 & 0 & 0 & 1 & 35 \end{bmatrix}$$

We see that $T_1 = 65°$, $T_2 = 60°$, $T_3 = 40°$ and $T_4 = 35°$.

NTNU
Norwegian University of
Science and Technology

Linear independence

An indexed set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be *linearly independent* if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution.

NTNU
Norwegian University of
Science and Technology

Linear dependence

An indexed set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be *linearly dependent* if there exist weights $c_1, c_2, \ldots, c_p$ not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}.$$

NTNU
Norwegian University of
Science and Technology

Example

Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

Let us determine if the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent or linearly independent.

We must determine if the equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$$

has a nontrivial solution or not.

We write down the augmented matrix $[\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{0}]$ of the equation and reduce it to echelon form.

NTNU
Norwegian University of
Science and Technology

Solution

$$[\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{0}] = \begin{bmatrix} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & -6 & -6 & 0 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We see that the system has a free variable, so $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent.

NTNU
Norwegian University of
Science and Technology

Linear independence of matrix columns

Suppose that we begin with a matrix $A = [\mathbf{a}_1 \ \dots \ \mathbf{a}_n]$ instead of a set of vectors. The matrix equation $A\mathbf{x} = \mathbf{0}$ can be written as

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{0}.$$

Thus we have the following important fact.

Linear independence of matrix columns

The columns of a matrix A are linearly independent if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

NTNU
Norwegian University of
Science and Technology

Example

Let us determine if the columns of the matrix

$$A = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 5 & 8 & 0 \end{bmatrix}$$ are linearly independent.

We must determine if the equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution or not. We write down the augmented matrix $[A\ \mathbf{0}]$ of the equation and reduce it to echelon form.

NTNU
Norwegian University of
Science and Technology

Solution

$$[A\,\mathbf{0}] = \begin{bmatrix} 0 & 1 & 4 & 0 \\ 1 & 2 & -1 & 0 \\ 5 & 8 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 5 & 8 & 0 & 0 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & -2 & 5 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 13 & 0 \end{bmatrix}$$

We see that there are no free variables so $\mathbf{x} = 0$ is the only solution and the columns of A are therefore linearly independent.

NTNU
Norwegian University of
Science and Technology

Linear dependence of sets containing the zero vector

Theorem 9

If a set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ contains the zero vector, then the set is linearly dependent.

Proof: If $\mathbf{v}_j = \mathbf{0}$, then $c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$ when $c_j = 1$ and $c_i = 0$ for $i \neq j$. This shows that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is linearly dependent.

NTNU
Norwegian University of
Science and Technology

Sets of one vector

A set containing only one vector $\mathbf{v}$, is linearly independent if and only if $\mathbf{v}$ is not the zero vector.

NTNU
Norwegian University of
Science and Technology

Sets of two vectors

A set of two vectors is linearly independent if and only if neither of the vectors is a multiple of the other.

NTNU
Norwegian University of
Science and Technology

Sets of two or more vectors

Theorem 7: Characterization of linearly dependent sets

An indexed set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others.
In fact, if S is linearly dependent and $\mathbf{v}_1 \neq 0$, then some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{j-1}$.

NTNU
Norwegian University of Science and Technology

Proof of Theorem 7

Suppose there exist an index j and scalars $c_1, \ldots, c_{j-1}, c_{j+1}, \ldots, c_p$ such that

$$\mathbf{v}_j = c_1\mathbf{v}_1 + \ldots c_{j-1}\mathbf{v}_{j-1} + c_{j+1}\mathbf{v}_{j+1} + \ldots c_p\mathbf{v}_p.$$

If we let $c_j = -1$, then

$$c_1\mathbf{v}_1 + \ldots c_j\mathbf{v}_j = \mathbf{0}$$

and $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is linearly dependent.

NTNU
Norwegian University of
Science and Technology

Proof of Theorem 7 (cont.)

Assume then that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is linearly dependent. If $\mathbf{v}_1 = \mathbf{0}$, then $\mathbf{v}_1$ is a linear combination of $\mathbf{v}_2, \ldots, \mathbf{v}_p$. Suppose $\mathbf{v}_1 \neq \mathbf{0}$. Since $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is linearly dependent, there exist scalars $c_1, \ldots, c_p$ which are not all equal to zero such that $c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$. Let j be the largest subscript such that $c_j \neq 0$. Notice that $j > 1$ because otherwise $c_1\mathbf{v}_1 = \mathbf{0}$ which cannot be the case since $\mathbf{v}_1 \neq \mathbf{0}$ and $c_1 \neq 0$. We then have that $c_1\mathbf{v}_1 + \cdots + c_j\mathbf{v}_j = \mathbf{0}$ from which it follows that $\mathbf{v}_j = \frac{-c_1}{c_j}\mathbf{v}_1 + \frac{-c_2}{c_j}\mathbf{v}_2 \cdots + \frac{-c_{j-1}}{c_j}\mathbf{v}_{j-1}$. Thus, $\mathbf{v}_j$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{j-1}$.

NTNU
Norwegian University of
Science and Technology

Sets of two or more vectors

- Theorem 7 does *not* say that *every* vector in a linearly dependent set is a linear combination of the preceding vectors.
- A vector in a linearly dependent set may fail to be a linear combination of the other vectors.

Example

Let $\mathbf{u} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix}$, and let $\mathbf{w}$ be an unknown vector in $\mathbb{R}^3$.

- If $\mathbf{w}$ is in Span$\{\mathbf{u}, \mathbf{v}\}$, then $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent because $\mathbf{w}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$.
- If $\mathbf{w}$ is not in Span$\{\mathbf{u}, \mathbf{v}\}$, then $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent because
  1. $\mathbf{u}$ is not the zero vector,
  2. $\mathbf{v}$ is not a linear combination of $\mathbf{u}$ ($\mathbf{v}$ is not a multiple of $\mathbf{u}$),
  3. $\mathbf{w}$ is not a linear combination of $\mathbf{u}$ and $\mathbf{v}$ because $\mathbf{w}$ is not in Span$\{\mathbf{u}, \mathbf{v}\}$.

NTNU
Norwegian University of
Science and Technology

Sets of two or more vectors

Theorem 8

If a set contains more vectors than there are entries in each vector, then the set is linearly dependent.
That is, any set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.

Proof: Let $\mathbf{v}_1, \ldots, \mathbf{v}_p$ be vectors in $\mathbb{R}^n$ and assume that $p > n$. Let A be the $n \times p$ matrix $[\mathbf{v}_1 \ \ldots \ \mathbf{v}_p]$. Since $p > n$, not every column of A can be a pivot column, so there must be free variables and the equation $A\mathbf{x} = \mathbf{0}$ has therefore nontrivial solutions. It follows that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is linearly dependent.

NTNU
Norwegian University of
Science and Technology

Example

The following sets are all linearly dependent:

1. $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ where $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, and $\mathbf{u}_3 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$, because $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ consists of 3 vectors in $\mathbb{R}^2$ and $3 > 2$.
2. $\{\mathbf{v}_1, \mathbf{v}_2\}$ where $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, because $\mathbf{v}_2 = \mathbf{0}$.

NTNU
Norwegian University of
Science and Technology

Example

3. $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ where $\mathbf{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, $\mathbf{w}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, and $\mathbf{w}_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$, because $\mathbf{w}_1 + \mathbf{w}_2 = \mathbf{w}_3$.

NTNU
Norwegian University of
Science and Technology

Problem 6 from August 2010

For which values of the parameter a are the vectors $\mathbf{v}_1 = (1, -3, a)$, $\mathbf{v}_2 = (0, 1, a)$ and $\mathbf{v}_3 = (a, 2, 0)$ linearly dependent?

Solution

The vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly dependent, if and only if the equation

$$\begin{bmatrix} 1 & 0 & a \\ -3 & 1 & 2 \\ a & a & 0 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

has a nontrivial solution.

We reduce the augmented matrix $\begin{bmatrix} 1 & 0 & a & 0 \\ -3 & 1 & 2 & 0 \\ a & a & 0 & 0 \end{bmatrix}$ to an

echelon form.

Solution (cont.)

$$\begin{bmatrix} 1 & 0 & a & 0 \\ -3 & 1 & 2 & 0 \\ a & a & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & a & 0 \\ 0 & 1 & 2+3a & 0 \\ 0 & a & -a^2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & a & 0 \\ 0 & 1 & 2+3a & 0 \\ 0 & 0 & -2a-4a^2 & 0 \end{bmatrix}$$

We see that $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly dependent if and only if $-2a-4a^2 = -2a(1+2a) = 0$, that is, if and only if $a = 0$ or $a = -1/2$.

Problem 5 from December 2007

Suppose that $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are linearly independent vectors in $\mathbb{R}^n$. Are vectors

$$\mathbf{w}_1 = \mathbf{v}_1 + \mathbf{v}_2,\ \mathbf{w}_2 = \mathbf{v}_1 + \mathbf{v}_3,\ \mathbf{w}_3 = \mathbf{v}_2 + \mathbf{v}_3$$

linearly dependent or linearly independent? (Remember to explain the answers.)

NTNU
Norwegian University of
Science and Technology

Solution

Suppose that

$$c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3 = \mathbf{0}.$$

Then

$$\begin{aligned}\mathbf{0} &= c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3 \\ &= c_1(\mathbf{v}_1 + \mathbf{v}_2) + c_2(\mathbf{v}_1 + \mathbf{v}_3) + c_3(\mathbf{v}_2 + \mathbf{v}_3) \\ &= (c_1 + c_2)\mathbf{v}_1 + (c_1 + c_3)\mathbf{v}_2 + (c_2 + c_3)\mathbf{v}_3.\end{aligned}$$

It follows from the linear independence of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ that $c_1 + c_2 = c_1 + c_3 = c_2 + c_3 = 0$. The only solution of $c_1 + c_2 = c_1 + c_3 = c_2 + c_3 = 0$ is $c_1 = c_2 = c_3$, so if $c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3 = \mathbf{0}$, then $c_1 = c_2 = c_3$. This shows that $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_3$ are linearly independent.

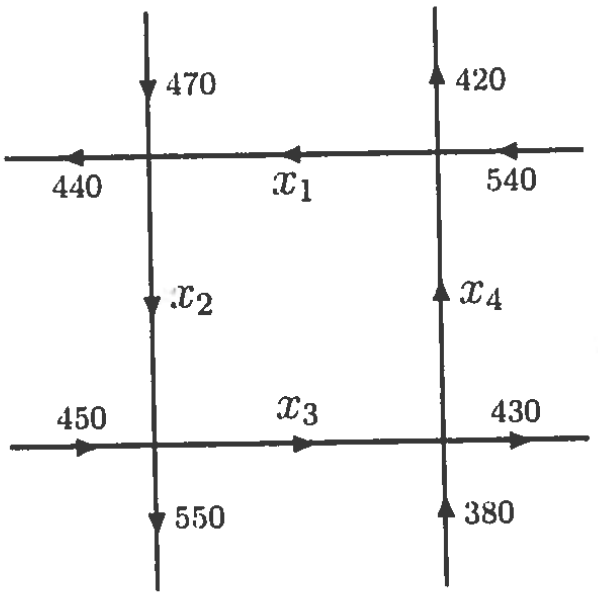
NTNU
Norwegian University of
Science and Technology

Problem 4 from December 2006

Problem 4 Four one-way streets in a city intesect as shown in the figure to the right. The number of cars passing per hour is shown in the figure.

Write down a system of equations in the form $A\mathbf{x} = \mathbf{b}$ which $\mathbf{x} = (x_1, x_2, x_3, x_4)$ must satisfy, and solve the system.

Find x_1, x_2 and x_3 if the x_4-section is closed for traffic, making $x_4 = 0$.



NTNU
Norwegian University of
Science and Technology

Solution

By looking at the number of cars going in and out of each of the 4 intersections, we see that the following equations must be satisfied:

$$
\begin{aligned}
x_1 + 470 &= 440 + x_2 \\
x_2 + 450 &= 550 + x_3 \\
x_3 + 380 &= 430 + x_4 \\
x_4 + 540 &= 420 + x_1
\end{aligned}
$$

NTNU
Norwegian University of
Science and Technology

Solution (cont.)

This system of equations is equivalent to the equation $A\mathbf{x} = \mathbf{b}$ where $A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -30 \\ 100 \\ 50 \\ -120 \end{bmatrix}$.

We solve the system by reducing the augmented matrix $[A\ \mathbf{b}]$ to its reduced echelon form.

$$\begin{bmatrix} 1 & -1 & 0 & 0 & -30 \\ 0 & 1 & -1 & 0 & 100 \\ 0 & 0 & 1 & -1 & 50 \\ -1 & 0 & 0 & 1 & -120 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 & 0 & -30 \\ 0 & 1 & -1 & 0 & 100 \\ 0 & 0 & 1 & -1 & 50 \\ 0 & -1 & 0 & 1 & -150 \end{bmatrix}$$

NTNU
Norwegian University of
Science and Technology

Solution (cont.)

$$\to \begin{bmatrix} 1 & -1 & 0 & 0 & -30 \\ 0 & 1 & -1 & 0 & 100 \\ 0 & 0 & 1 & -1 & 50 \\ 0 & 0 & -1 & 1 & -50 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & 0 & -30 \\ 0 & 1 & -1 & 0 & 100 \\ 0 & 0 & 1 & -1 & 50 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -1 & 0 & 0 & -30 \\ 0 & 1 & 0 & -1 & 150 \\ 0 & 0 & 1 & -1 & 50 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1 & 120 \\ 0 & 1 & 0 & -1 & 150 \\ 0 & 0 & 1 & -1 & 50 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

NTNU
Norwegian University of
Science and Technology

Solution (cont.)

We see that the system is consistent and that the general solution of it is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 120 \\ 150 \\ 50 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

where t is a free parameter.
If $x_4 = 0$, then $x_1 = 120$, $x_2 = 150$ and $x_3 = 50$.

NTNU
Norwegian University of
Science and Technology

Problem 6 from August 2005

Let A be an $m \times n$ matrix where $m > n$. Explain why there is a vector $\mathbf{b}$ in $\mathbb{R}^m$ such that the equation $A\mathbf{x} = \mathbf{b}$ does not have a solution.

NTNU
Norwegian University of
Science and Technology

Solution

The equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b} \in \mathbb{R}^m$ if and only if A has a pivot position in every row. Since $m > n$, A has more rows that columns, and it can therefore not have a pivot position in every row (since the pivot positions appear in different columns). Thus, there is a vector $\mathbf{b} \in \mathbb{R}^m$ such that the equation $A\mathbf{x} = \mathbf{b}$ does not have a solution.

NTNU
Norwegian University of
Science and Technology

Plan for next week

Wednesday we shall introduce and study

- *linear transformations*,
- the *standard matrix* of a linear transformation,
- *onto* linear transformations,
- *one-to-one* linear transformations,

Section 1.8-1.9 in “Linear Algebras and Its Applications” (pages 62-80).

Thursday we shall look at *applications of linear models*. Section 1.10 in “Linear Algebras and Its Applications” (pages 80-90).

NTNU
Norwegian University of
Science and Technology