Norwegian University of Science
and Technology
Department of Mathematical
Sciences

TMA4115 Calculus 3
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Solutions to exercise set 11

1 Describe all possible least-squares solutions of the system

$$x - 2y = 0,$$
$$x - 2y = 1.$$

By definition, a least-square solution of the system is a vector $\left[\begin{smallmatrix}\hat{x}\\ \hat{y}\end{smallmatrix}\right]$ which minimizes $(\hat{x}-2\hat{y}-0)^2+(\hat{x}-2\hat{y}-1)^2$. Letting $\hat{b} = \hat{x}-2\hat{y}$ we get that $(\hat{x}-2\hat{y}-0)^2+(\hat{x}-2\hat{y}-1)^2 = \hat{b}^2 + (\hat{b}-1)^2 = 2\hat{b}^2 - 2\hat{b} + 1$. This quadratic function obtains its minimum when $\hat{b} = \frac{1}{2}$, so a least squares solution of the original system is any $(\hat{x}, \hat{y})$ which satisfies

$$\hat{x} - 2\hat{y} = \frac{1}{2}$$

2 An experiment carried out to investigate the dependence of a quantity y on the independent variable x produces the data points

$$(0,-1),\ (1,-1),\ (2,0),\ (3,1).$$

Find the least squares line $y = \beta_0 + \beta_1 x$ that best fits the above data.

Here, the design matrix X, the parameter vector $\boldsymbol{\beta}$, and the observatrion vector $\mathbf{y}$, are, respectively

$$X = \begin{bmatrix}1 & 0\\ 1 & 1\\ 1 & 2\\ 1 & 3\end{bmatrix}, \qquad \boldsymbol{\beta} = \begin{bmatrix}\beta_0\\ \beta_1\end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix}-1\\ -1\\ 0\\ 1\end{bmatrix}.$$

The normal equations are $X^T X\boldsymbol{\beta} = X^T\mathbf{y}$, that is

$$\begin{bmatrix}4 & 6\\ 6 & 14\end{bmatrix}\begin{bmatrix}\beta_0\\ \beta_1\end{bmatrix} = \begin{bmatrix}-1\\ 2\end{bmatrix},$$

with solution $\beta_0 = -\frac{13}{10}, \quad \beta_1 = \frac{7}{10}$. Therefore, the least squares line is $y = -\frac{13}{10} + \frac{7}{10}x$.

3 Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix}.$$

Give the dimensions of the eigenspaces associated with each of the eigenvalues of A **without computing any of the eigenspaces.** Give reasons for your answer.

The characteristic polynomial of A is

$$\det(A - \lambda I) = (2 - \lambda)((3 - \lambda)^2 - 1) = -(\lambda - 2)(\lambda^2 - 6\lambda + 8).$$

The roots of the quadratic polynomial $\lambda^2 - 6\lambda + 8$ are 2 and 4. Therefore, the eigenvalues of A are 2, with algebraic multiplicity[1] 2, and 4, with algebraic multiplicity 1. A is symmetric, so by the spectral theorem for symmetric matrixes, we can immediately conclude that the eigenspace corresponding to the eigenvalue 2 has dimension 2, and that the eigenspace corresponding to the eigenvalue 4 has dimension 1.

[1] The *algebraic multiplicity* of an eigenvalue is its multiplicity as a root of the characteristic polynomial.