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TMA4115 Calculus 3
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Solutions to exercise set 11

1 Let $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 8 \\ 1 \end{bmatrix}$. Write $\mathbf{v}$ as a sum of a vector in $\text{Span}\{\mathbf{u}\}$ and a vector orthogonal to $\mathbf{u}$.

The orthogonal projection of $\mathbf{v}$ on $\text{Span}\{\mathbf{u}\}$ is

$$\hat{\mathbf{v}} = \frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u} = \frac{10}{5}\mathbf{u} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Now $\mathbf{v} - \hat{\mathbf{v}} = \begin{bmatrix} 6 \\ -3 \end{bmatrix}$, is orthogonal to $\mathbf{u}$, so $\mathbf{v} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 6 \\ -3 \end{bmatrix}$ solves the problem.

2 Let $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$. Find the distance of $\mathbf{v}$ from the plane in $\mathbb{R}^3$ spanned by $\mathbf{u}_1$ and $\mathbf{u}_2$.

$\mathbf{u}_1$ and $\mathbf{u}_2$ are orthogonal, so the orthogonal projection of $\mathbf{v}$ on $\text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$ is given by

$$\hat{\mathbf{v}} = \frac{\mathbf{u}_1\cdot\mathbf{v}}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{u}_2\cdot\mathbf{v}}{\mathbf{u}_2\cdot\mathbf{u}_2}\mathbf{u}_2 = \frac{12}{6}\mathbf{u}_1 + \frac{(-4)}{2}\mathbf{u}_2 = 2\mathbf{u}_1 - 2\mathbf{u}_2 = \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix},$$

and $\mathbf{v} - \hat{\mathbf{v}} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. The distance from $\mathbf{v}$ to $\text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$ is given by
$\|\mathbf{v} - \hat{\mathbf{v}}\| = \sqrt{(\mathbf{v} - \hat{\mathbf{v}})\cdot(\mathbf{v} - \hat{\mathbf{v}})} = \sqrt{3}$.

3 Apply the Gram–Schmidt process to the vectors

$$\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix},$$

to find an orthonormal basis for $\mathbb{R}^3$. Find the coordinates of $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ relative to this orthonormal basis.

Let

$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \ \mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \ \mathbf{x}_3 = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix},$$

and apply the Gram–Schmidt process as described in Theorem 11, p. 355 in the

textbook. We obtain

$$\mathbf{v}_1 = \mathbf{x}_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix},$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \mathbf{x}_2 - \frac{3}{9}\mathbf{v}_1 = \frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix},$$

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 = \mathbf{x}_3 - \frac{12}{9}\mathbf{v}_1 - 0\mathbf{v}_2 = \frac{1}{3}\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}.$$

$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ form an *orthogonal* basis for $\mathbb{R}^3$. To obtain an *orthonormal* basis, we have to scale each of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ by the inverse of their norm. We see that

$$\|\mathbf{v}_1\| = 3 \quad \|\mathbf{v}_2\| = 1 \quad \|\mathbf{v}_3\| = 1,$$

So an orthonormal basis of $\mathbb{R}^3$ is

$$\mathcal{B} = \left\{ \mathbf{u}_1 = \begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix}, \ \mathbf{u}_2 = \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix}, \ \mathbf{u}_3 = \begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \end{bmatrix} \right\}.$$

To obtain the coordinates of $\mathbf{y} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ relative to this basis, we only need to calculate

$$\mathbf{u}_1 \cdot \mathbf{y} = \frac{11}{3} \quad \mathbf{u}_2 \cdot \mathbf{y} = \frac{1}{3} \quad \mathbf{u}_3 \cdot \mathbf{y} = \frac{-2}{3}$$

So $[\mathbf{y}]_{\mathcal{B}} = \begin{bmatrix} 11/3 \\ 1/3 \\ -2/3 \end{bmatrix}$.