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TMA4115 Calculus 3
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Solutions to exercise set 11

1 In this problem, b is some positive real number. Diagonalize the matrix $A = \begin{bmatrix} 0 & 1 \\ b & (b-1) \end{bmatrix}$, that is, find (and explicitly write down) an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

The eigenvalues of A are the solutions to the polynomial equation $\det(A - \lambda I) = \lambda^2 - (b-1) - b = 0$. The polynomial has solutions $\lambda_1 = b, \lambda_2 = -1$.

The eigenvector corresponding to the eigenvalue b is $\left[\begin{smallmatrix} 1 \\ b \end{smallmatrix}\right]$, while the eigenvector corresponding to the eigenvalue -1 is $\left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]$.

Therefore $A = PDP^{-1}$ with $P = \begin{bmatrix} 1 & b \\ 1 & -1 \end{bmatrix}$ and $D = \begin{bmatrix} b & 0 \\ 0 & -1 \end{bmatrix}$.

This answer is not unique, and other answers may also be correct.

2 Find the eigenvalues and a basis for each eigenspace in $\mathbb{C}^2$ for the matrix $A = \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}$.

The characteristic polynomial is $\det(A - \lambda I) = \lambda^2 - 2\lambda + 10$, whose roots are $\lambda = 1 \pm 3i$. The eigenspace corresponding to $\lambda_1 = 1 + 3i$ is the nullspace of $A - (1 + 3i)I = \begin{bmatrix} -3i & 3 \\ -3 & -3i \end{bmatrix}$. We see that $\mathbf{v}_1 \left[\begin{smallmatrix} 1 \\ i \end{smallmatrix}\right]$ is a basis for this space. The eigenvector corresponding to $\lambda_2 = 1 - 3i = \overline{\lambda}_1$ can be found easily. A is real, so

$$A\overline{\mathbf{v}}_1 = \overline{A\mathbf{v}_1} = \overline{\lambda_1 \mathbf{v}_1} = \lambda_2 \overline{\mathbf{v}}_1.$$

Therefore, $\mathbf{v}_2 = \overline{\mathbf{v}}_1 = \left[\begin{smallmatrix} 1 \\ -i \end{smallmatrix}\right]$ is an eigenvector corresponding to $1 - 3i$. The eigenspace has to be one dimensional, so $\mathbf{v}_2$ is also a basis for this eigenspace.

3 You are given the initial value problem:

$$4y'' + 4y' + y = 0, \quad y(0) = 0,\ y'(0) = -1.$$

Transform this into an initial-value problem for a first-order *system* of two linear differential equations in the unknowns x_1 and x_2.

If we set $x_1 = y, x_2 = y'$, then

$$\begin{aligned} x_1' &= y' = x_2 \\ x_2' &= y'' = -\frac{1}{4}y - y' = -\frac{1}{4}x_1 - x_2 \end{aligned}$$

The initial values are read directly

$$x_1(0) = y(0) = 0, \qquad x_2(0) = y'(0) = -1$$