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TMA4115 Calculus 3
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Solutions to exercise set 10

1 Find bases for $\mathrm{Col}(A)$, $\mathrm{Row}(A)$ and $\mathrm{Nul}(A)$ for the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 2 & 2 & 3 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}.$$

By row reduction, we see that A is row-equivalent to

$$B = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The pivot columns are the first, second and fifth columns. A basis for $\mathrm{Col}(A)$ is given by the corresponding columns in A, that is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 3 \\ 5 \end{bmatrix} \right\}.$$

A basis for $\mathrm{Row}(A)$ is given by the non-zero rows of B, that is

$$\left\{ \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right\}.$$

Finally, we see that the general solution to $A\mathbf{x} = \mathbf{0}$ is

$$\mathbf{x} = s \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix},$$

so a basis for $\mathrm{Nul}(A)$ is

$$\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Note that the basis for a given subspace is not unique, and other answers may also be correct.

2 Let A be a 7×4 matrix. Suppose $A\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \mathbf{0}$. What is the largest possible dimension of $\mathrm{Row}(A)$? Give reasons for your answer.

By the rank theorem, we know that $\dim \mathrm{Row}(A) = \mathrm{rank}(A)$ and that $\mathrm{rank}(A) + \dim \mathrm{Nul}(A) = 4$. From the statement of the problem, $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \in \mathrm{Nul}(A)$, so the dimension of $\mathrm{Nul}(A)$ is at least one. Therefore $\dim \mathrm{Row}(A) \leq 4-1 = 3$. To show that $\dim \mathrm{Row}(A) = \mathrm{rank}(A) = 3$ is possible under the restrictions in the problem, choose three linearly independent vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ in $\mathbb{R}^7$ and let $A = [\mathbf{v}_1, \mathbf{v}_2, -\mathbf{v}_3, \mathbf{v}_4]$.

3 The gates of a dam on a river are, on any one day, either raised for half a day to discharge water downstream or kept shut. It has been observed over many years that if the gates are raised on one day then there is a 90% chance that they will stay shut the next day, and that if they stay shut then there is a 20% chance that they will stay shut the next day. According to this information, what is the probability that the gates of the dam will stay shut on a given day?

If we define a sequence of vectors $\mathbf{x}_k = \begin{bmatrix} p_k \\ q_k \end{bmatrix}$, $k = 1, 2, \ldots$, where

$$p_k = \text{probability that the gates stay shut on day } k,$$
$$q_k = \text{probability that the gates are opened on day } k,$$

then the sequence of vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ form a Markov chain with stochastic matrix

$$P = \begin{bmatrix} 0.2 & 0.9 \\ 0.8 & 0.1 \end{bmatrix}$$

The steady state of the system is the probability vector which solves the set of linear equations $(P - I)\mathbf{x} = \mathbf{0}$. The general solution to $(P - I)\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = s\begin{bmatrix} 9 \\ 8 \end{bmatrix}$. For $\mathbf{x}$ to be a probability vector, we need $9s + 8s = 1$, so we set $s = \frac{1}{17}$ and

$$\mathbf{x} = \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} \frac{9}{17} \\ \frac{8}{17} \end{bmatrix}.$$

The probability that the gates stay shut a given day is $p = \frac{9}{17}$.