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TMA4115 Calculus 3
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Solutions to exercise set 9

1 Let W be the set of all vectors of the form $\begin{bmatrix} 3a-5b \\ 3b+2a \\ 1 \end{bmatrix}$, where a and b represent arbitrary real numbers. Determine if W is a subspace of $\mathbb{R}^3$.

$\mathbf{0}$ is not in W, so W is not a subspace of $\mathbb{R}^3$.

2 Find the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ of the vector $\mathbf{x}$ with respect to the basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$, where:

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix}, \quad \mathbf{b}_3 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \text{ and } \mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}.$$

By inspection (or by solving the linear system of equations), we see that

$$\mathbf{x} = \mathbf{b}_1 - \mathbf{b}_2 + \mathbf{b}_3.$$

Therefore

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

3 Let $A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 4 & 8 & 2 & 1 \end{bmatrix}$. Find a basis for the null space $\text{Nul}(A)$ of A.

We use the algorithm outlined in example 3, page 200-201 in the text book. The reduced echelon form of A is $B = \begin{bmatrix} 1 & 2 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} \end{bmatrix}$, and the general solution to the homogeneous solution $A\mathbf{x} = \mathbf{0}$ is given in terms of the free variables x_2 and x_4 as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_2 - \frac{1}{2}x_4 \\ x_2 \\ \frac{1}{2}x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

So a basis for $\text{Nul}(A)$ is

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 1 \end{bmatrix} \right\}$$

4 Let $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ and $\mathbf{p}_4$ be the elements of $\mathbb{P}_3$ (the vector space of polynomials of degree ≤ 3 with real coefficients) given by $\mathbf{p}_1(x) = 1$, $\mathbf{p}_2(x) = 2x^2 - 1$, $\mathbf{p}_3(x) = 4x^3 - 3x$ and $\mathbf{p}_4(x) = -4x^3 + 4x^2 + 3x - 1$. Is $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4\}$ a basis for $\text{Span}\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4\}$? Give a reason for your answer.

$\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4\}$ clearly span $\text{Span}\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4\}$, so they are a basis if and only if they are linearly independent. We can see, by inspection or by solving a linear system, that

$$-\mathbf{p}_1 - 2\mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4 = 0$$

as polynomials. That is

$$-\mathbf{p}_1(x) - 2\mathbf{p}_2(x) + \mathbf{p}_3(x) + \mathbf{p}_4(x) = 0$$

for all x. The polynomials are linearly dependent, and are therefore not a basis for $\text{Span}\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4\}$.