Norwegian University of Science
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TMA4115 Calculus 3
Spring 2013

Solutions to exercise set 8

1 Compute the determinant

$$\begin{vmatrix} 1 & 0 & -1 & -2 \\ 0 & -1 & -2 & 1 \\ 2 & 0 & 2 & 0 \\ 7 & 0 & -5 & -1 \end{vmatrix}$$

using cofactor expansions. At each step, choose a row or column that involves the least amount of computation.

At the first step, we expand along the second column,

$$\begin{vmatrix} 1 & 0 & -1 & -2 \\ 0 & -1 & -2 & 1 \\ 2 & 0 & 2 & 0 \\ 7 & 0 & -5 & -1 \end{vmatrix} = -1 \begin{vmatrix} 1 & -1 & -2 \\ 2 & 2 & 0 \\ 7 & -5 & -1 \end{vmatrix}$$

We can now expand along the second row of the remaining matrix to calculate

$$\begin{aligned} -1 \begin{vmatrix} 1 & -1 & -2 \\ 2 & 2 & 0 \\ 7 & -5 & -1 \end{vmatrix} &= -1 \left(-2 \begin{vmatrix} -1 & -2 \\ -5 & -1 \end{vmatrix} + 2 \begin{vmatrix} 1 & -2 \\ 7 & -1 \end{vmatrix} \right) \\ &= -1(-2(1-10) + 2(-1-(-14))) \\ &= -44 \end{aligned}$$

2 Let A and B be two $n \times n$ matrices, $n \geq 2$, with $\det A = 5$ and $\det B = 2$. Use the properties of determinants to compute $\det(A^T B^3 A)$.

By muliplicativity of the determinant and $\det(A^T) = \det A$, we get

$$\det(A^T B^3 A) = \det(A^T) \det(B^3) \det(A) = \det(A) \det(B)^3 \det(A) = 5 \cdot 2^3 \cdot 5 = 200$$

3 You are given that

$$\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} = 3.$$

Use this information to compute the determinant

$$\begin{vmatrix} x & y & z \\ p & q & r \\ 3a+4x & 3b+4y & 3c+4z \end{vmatrix}.$$

By inspection, we see that we can obtain the first matrix from the second matrix by

a series of elementary row operations (or vice versa.) By using the properties listed in theorem 3, p. 169 in the textbook,

$$\begin{vmatrix} x & y & z \\ p & q & r \\ 3a+4x & 3b+4y & 3c+4z \end{vmatrix} = \begin{vmatrix} x & y & z \\ p & q & r \\ 3a & 3b & 3c \end{vmatrix} \quad \text{adding a multiple of one row to another}$$

$$= 3 \cdot \begin{vmatrix} x & y & z \\ p & q & r \\ a & b & c \end{vmatrix} \quad \text{multiplication of a row with a constant}$$

$$= -3 \cdot \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} \quad \text{interchanging two rows}$$

$$= -3 \cdot 3 = -9.$$

4 Find all values of the parameter $s \in \mathbb{R}$ for which the system of equations

$$sx_1 + 2x_2 = 7$$
$$3sx_1 + 2sx_2 = 1$$

has a *unique* solution. Use Cramer's rule to find this solution (which will be in terms of s).

The system of equations can be written on the form $A(s)\mathbf{x} = \mathbf{b}$, where

$$A(s) = \begin{bmatrix} s & 2 \\ 3s & 2s \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

$\det(A(s)) = 2s^2 - 6s$, and the system of equations will have a unique solution iff $A(s) \neq 0$ that is to say, when s is not equal to 0 or 3. By Cramer's rule

$$x_1 = \frac{\begin{vmatrix} \mathbf{b} & \mathbf{a}_2 \end{vmatrix}}{\det A} = \frac{\begin{vmatrix} 7 & 2 \\ 1 & 2s \end{vmatrix}}{\det A} = \frac{14s - 2}{2s^2 - 6s}$$

$$x_2 = \frac{\begin{vmatrix} \mathbf{a}_1 & \mathbf{b} \end{vmatrix}}{\det A} = \frac{\begin{vmatrix} s & 7 \\ 3s & 1 \end{vmatrix}}{\det A} = \frac{-20s}{2s^2 - 6s}$$