Norwegian University of Science
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TMA4115 Calculus 3
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Solutions to exercise set 5

1 For which values of h are the vectors $\begin{bmatrix}1\\-1\\2\end{bmatrix}$, $\begin{bmatrix}-1\\2\\0\end{bmatrix}$, $\begin{bmatrix}2\\h\\4\end{bmatrix}$ linearly dependent?

We call the vectors $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$, respectively. $\{\mathbf{v}_1, \mathbf{v}_2\}$ are linearly independent, so the three vectors are linearly dependent if and only if $\mathbf{v}_3$ can be written as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_3$. (Theorem 7, p.58 in the text book.)

We see that $\mathbf{v}_3 - 2\mathbf{v}_1 = \begin{bmatrix}0\\h+2\\0\end{bmatrix}$, and that adding a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ to this expression would make the first or third (or both) entries nonzero. Thus $\mathbf{v}_3$ can be written as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, specifically $\mathbf{v}_3 = 2\mathbf{v}_1$, if and only if $h = -2$. (Could also have used row reduction.)

2 Let $\mathbf{e}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$, $\mathbf{e}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$, $\mathbf{y}_1 = \begin{bmatrix}2\\3\end{bmatrix}$, $\mathbf{y}_2 = \begin{bmatrix}0\\0\end{bmatrix}$, and $\mathbf{y}_3 = \begin{bmatrix}-1\\4\end{bmatrix}$, and let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation such that $T(\mathbf{e}_1) = \mathbf{y}_1$, $T(\mathbf{e}_2) = \mathbf{y}_2$, and $T(\mathbf{e}_3) = \mathbf{y}_3$. Find $T\left(\begin{bmatrix}3\\1\\-1\end{bmatrix}\right)$.

We use the properties of linear transforms,

$$\begin{aligned} T\left(\begin{bmatrix}3\\1\\-1\end{bmatrix}\right) &= T\left(3\mathbf{e}_1 + 1\mathbf{e}_2 - 1\mathbf{e}_3\right) \\ &= 3T(\mathbf{e}_1) + T(\mathbf{e}_2) - T(\mathbf{e}_3) \\ &= 3\mathbf{y}_1 + \mathbf{y}_2 - \mathbf{y}_3 \\ &= 3 \cdot \begin{bmatrix}2\\3\end{bmatrix} + \begin{bmatrix}0\\0\end{bmatrix} - \begin{bmatrix}-1\\4\end{bmatrix} \\ &= \begin{bmatrix}7\\5\end{bmatrix}. \end{aligned}$$

3 Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation such that $T(x_1, x_2, x_3) = (2x_2 + 2x_3, 3x_1 + 3x_2)$. Determine if T is one-to-one, and if T is onto.

By Theorem 12, p.77 in the text book, part b., a linear transform is one-to-one if and only if the columns of the matrix of the transform are linearly independent. The matrix of the linear transform T is a 2×3-matrix, so it has 3 columns and 2 entries in each column. Since there are more vectors than entries in each vector, the columns can not be linearly independent, and T is *not one-to-one.*

By definition, T is onto if each $\mathbf{b}$ in $\mathbb{R}^2$ is the image of at least one $\mathbf{x}$ in $\mathbb{R}^3$. We see that for arbitrary s and t,

$$T\left(\frac{1}{3}t, 0, \frac{1}{2}s\right) = (s, t),$$

so T is *onto.*