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TMA4115 Calculus 3
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Solutions to exercise set 5

1 Let $\mathbf{v}_1 = \begin{bmatrix}0\\2\\1\end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix}1\\1\\2\end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix}1\\1\\1\end{bmatrix}$. Determine if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ span $\mathbb{R}^3$.

By Theorem 4, page 37 in the textbook, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ span $\mathbb{R}^3$ if the matrix A with $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ as columns has a pivot element in each row.

$$A = \begin{bmatrix}0 & 1 & 1\\2 & 1 & 1\\1 & 2 & 1\end{bmatrix}$$

We row reduce A to echelon form

$$\begin{bmatrix}0 & 1 & 1\\2 & 1 & 1\\1 & 2 & 1\end{bmatrix} \xrightarrow{\text{swap}(R_1,R_3)} \begin{bmatrix}1 & 2 & 1\\2 & 1 & 1\\0 & 1 & 1\end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{bmatrix}1 & 2 & 1\\0 & -3 & -1\\0 & 1 & 1\end{bmatrix} \xrightarrow{\text{swap}(R_2,R_3)}$$
$$\begin{bmatrix}1 & 2 & 1\\0 & 1 & 1\\0 & -3 & -1\end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 + 3R_1} \begin{bmatrix}1 & 2 & 1\\0 & 1 & 1\\0 & 0 & 2\end{bmatrix}$$

We see that no row of the form $[0 \quad 0 \quad 0]$ occur, and we can conclude that there is a pivot element in each row, and that $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \mathbb{R}^3$.

2 Let $A = \begin{bmatrix}1 & 2 & 0 & -3\\-2 & -4 & 1 & 6\\1 & 2 & 1 & -3\end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix}7\\-11\\10\end{bmatrix}$. Describe the solution set of the equation $A\mathbf{x} = \mathbf{b}$ in parametric vector form.

We write the augmented matrix of the equation and row reduce

$$\begin{bmatrix}1 & 2 & 0 & -3 & 7\\-2 & -4 & 1 & 6 & -11\\1 & 2 & 1 & -3 & 10\end{bmatrix} \xrightarrow[R_3 \leftarrow R_3 - R_1]{R_2 \leftarrow R_2 + 2R_1} \begin{bmatrix}1 & 2 & 0 & -3 & 7\\0 & 0 & 1 & 0 & 3\\0 & 0 & 1 & 0 & 3\end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 - R_2} \begin{bmatrix}1 & 2 & 0 & -3 & 7\\0 & 0 & 1 & 0 & 3\\0 & 0 & 0 & 0 & 0\end{bmatrix}$$

We see that x_2 and x_4 are free variables, and that the general solution is

$$x_1 = 7 - 2s + 3t \quad x_2 = s \quad x_3 = 3 \quad x_4 = t$$

Which we can rewrite in vector form as

$$\mathbf{x} = \begin{bmatrix}7\\0\\3\\0\end{bmatrix} + s\begin{bmatrix}-2\\1\\0\\0\end{bmatrix} + t\begin{bmatrix}3\\0\\0\\1\end{bmatrix}.$$

3

Balance the chemical equation

$$C_2H_6 + O_2 \to CO_2 + H_2O$$

That is, find for each of the 4 types of molecules the right number of molecules such that for each type of atoms, the number of that type of atoms that appear on the left side of the equation is equal to the number of atoms of the same type that appear on the right hand side of the equation.

Let the unknown number of molecules be x_1, x_2, x_3, x_4, that is, the balanced equation is of the form

$$x_1C_2H_6 + x_2O_2 \to x_3CO_2 + x_4H_2O$$

For each element, we get the following equations

$$\begin{array}{lrcrcr} \text{C:} & 2x_1 & = & x_3 & & \\ \text{H:} & 6x_1 & = & & & 2x_4 \\ \text{O:} & 2x_2 & = & 2x_3 & + & x_4 \end{array}$$

Which we can arrange to the coefficient matrix

$$\begin{bmatrix} 2 & 0 & -1 & 0 \\ 6 & 0 & 0 & -2 \\ 0 & 2 & -2 & -1 \end{bmatrix}$$

(We could also have formed the augmented matrix, since the rightmost column of the augmented matrix is all zeros, we don't bother writing it up.) We row reduce

$$\begin{bmatrix} 2 & 0 & -1 & 0 \\ 6 & 0 & 0 & -2 \\ 0 & 2 & -2 & -1 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - 3R_1} \begin{bmatrix} 2 & 0 & -1 & 0 \\ 0 & 0 & 3 & -2 \\ 0 & 2 & -2 & -1 \end{bmatrix} \xrightarrow{\text{swap}(R_2, R_3)} \begin{bmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & -2 & -1 \\ 0 & 0 & 3 & -2 \end{bmatrix}$$

The matrix is in echelon form, and we can start solving by back-substitution. We see that x_4 is a free parameter, so we set $x_4 = t$ and get

$$x_3 = \frac{2}{3}t \quad x_2 = \frac{1}{2}(t + 2x_3) = \frac{7}{6}t \quad x_1 = \frac{1}{2}x_3 = \frac{1}{3}t$$

It is customary to present balanced chemical reaction with only integer values for x_1, x_2, x_3, x_4, so we set $t = 6$ and get the balanced equation

$$2C_2H_6 + 7O_2 \to 4CO_2 + 6H_2O.$$