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TMA4115 Calculus 3
Spring 2013

Solutions to exercise set 3

1 Find the general solution of the differential equation $y'' + y = \cos(t)$.

The corresponding homogenous equation is $y'' + y = 0$, with homogenous solution

$$y_h = a\cos t + b\sin t.$$

Since the right hand side is a solution to the homogenous equation, we search for a particular solution of the form

$$y_p = ct\cos t + dt\sin t.$$

Differentiating twice, we get

$$\begin{aligned} y_p' &= c\cos t - ct\sin t + d\sin t + dt\cos t \\ y_p'' &= -2c\sin t - ct\cos t + 2d\cos t - dt\sin t. \end{aligned}$$

Inserting y_p into the differential equation leads to

$$y_p'' + y_p = 2d\cos t - 2c\sin t = \cos t,$$

so $c = 0$, $d = \frac{1}{2}$, thus $y_p = \frac{1}{2}t\sin t$, and the general solution is

$$y = y_h + y_p = a\cos t + b\sin t + \frac{1}{2}t\sin t.$$

Here we could also have used the complex method.

2 Find a particular solution of the differential equation $y'' - 2y' + y = \frac{e^t}{t^2+1}$.

The homogenous equation is $y'' - 2y' + y = 0$, which has the two linearly independent solutions $y_1 = e^t$, $y_2 = te^t$ We use the method of variation of characteristics and write $y_p = v_1y_1 + v_2y_2$, now the equations for v_1, v_2 are

$$\begin{aligned} v_1'y_1 + v_2'y_2 &= (v_1' + tv_2')e^t = 0 \\ v_1'y_1' + v_2'y_2' &= (v_1' + (1+t)v_2')e^t = \frac{e^t}{t^2+1} \end{aligned}$$

which simplifies to

$$\begin{aligned} v_1' + tv_2' &= 0 \\ v_1' + (1+t)v_2' &= \frac{1}{t^2+1}. \end{aligned}$$

the solution is $v_1' = -\frac{t}{t^2+1}$, $v_2' = \frac{1}{t^2+1}$. By integration we obtain

$$v_1 = -\frac{1}{2}\ln(1+t^2), \quad v_2 = \tan^{-1}(t)$$

so

$$y_p = -\frac{1}{2}\ln(1+t^2)e^t + \tan^{-1}(t)te^t.$$

3 The motion of a mechanical system is described by the differential equation $y'' + 2y' + 5y = 2\sqrt{5}\cos(\sqrt{5}t)$. Determine whether the motion is under-damped, over-damped or critically damped and find the steady-state term of the motion.

The characteristic equation is $\lambda^2 + 2\lambda + 5 = 0$, with roots $\lambda = -1 \pm 2i$. This is an under-damped motion. The steady-state term is a particular solution of the non-homogenous system. Since the right hand side is $2\sqrt{5}\cos(\sqrt{5}t) = \mathrm{Re}(2\sqrt{5}e^{\sqrt{5}it})$ we can use the complex method. We look for a solution of $z'' + 2z' + 5z = 2\sqrt{5}e^{\sqrt{5}it}$ of the form $z_p = ae^{\sqrt{5}it}$. Inserting z_p into the equation gives

$$-5ae^{\sqrt{5}it} + 2\sqrt{5}iae^{\sqrt{5}it} + 5ae^{\sqrt{5}it} = 2\sqrt{5}e^{\sqrt{5}it}$$

or, after simplification $ia = 1 \Rightarrow a = \frac{1}{i} = -i$. So $z_p = -ie^{\sqrt{5}it}$ and

$$y_p = \mathrm{Re}\left(-ie^{\sqrt{5}it}\right) = \sin(\sqrt{5}t).$$