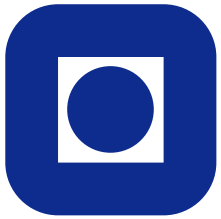

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TMA4115 - Calculus 3

Week 15

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Plan for today

This week we shall introduce and studied *eigenvalues*, *eigenvectors* and *eigenspaces*, the *characteristic equation* of a square matrix, and *diagonalization*.

- Complex eigenvalues (Section 5.5).
- Applications to differential equations (Section 5.7)
- Second-order equations and systems (section 4.2)

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Complex eigenvalues and eigenvectors

- A real matrix A might have a complex eigenvalue λ in which case the corresponding eigenvectors are also complex.
- If λ is an eigenvalue of A with corresponding eigenvector $\mathbf{v}$, then $\overline{\lambda}$ is an eigenvalue of A with corresponding eigenvector $\overline{\mathbf{v}}$.

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Real 2×2 matrices with complex eigenvalues

Theorem 9

Let A be a real 2×2 matrix with a complex eigenvalue $\lambda = a - bi$ ($b \neq 0$) and an associated eigenvector $\mathbf{v}$ in $\mathbb{C}^2$. Then

$$A = PCP^{-1}$$

where $P = [\mathrm{Re}(\mathbf{v}) \ \ \mathrm{Im}(\mathbf{v})]$ and $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

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Example

Let $A = \begin{bmatrix} 4 & -2 \\ 1 & 6 \end{bmatrix}$.

Rotation and scaling in $\mathbb{R}^2$

Let $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ where a and b are real numbers which are not both zero, and let $r = |a + ib| = \sqrt{a^2 + b^2}$ and $\phi = \text{Arg}(a + ib)$.

Then $C = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$ and the transformation $\mathbf{x} \mapsto C\mathbf{x}$ is the composition of a rotation through the angle ϕ and a scaling by r.

Systems of first order differential equations

A *system of first order differential equations with constant coefficients* is a system of differential equations

$$\begin{aligned} x_1' &= a_{11}x_1 + \cdots + a_{1n}x_n \\ x_2' &= a_{21}x_1 + \cdots + a_{2n}x_n \\ &\vdots \\ x_n' &= a_{n1}x_1 + \cdots + a_{nn}x_n \end{aligned}$$

where $x_1, \ldots, x_n$ are differentiable functions of t with derivatives $x_1', \ldots, x_n'$, and the a_{ij} are constants.

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Systems of first order differential equations

If we let

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} \quad \mathbf{x}'(t) = \begin{bmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{bmatrix} \text{ and } A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}$$

then we write the system as $\mathbf{x}'(t) = A\mathbf{x}(t)$.

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Systems of first order differential equations

- If $\mathbf{u}_1, \ldots, \mathbf{u}_k$ are solutions to the system $\mathbf{x}'(t) = A\mathbf{x}(t)$, then any linear combination $c_1\mathbf{u}_1 + \cdots + c_k\mathbf{u}_k$ of $\mathbf{u}_1, \ldots, \mathbf{u}_k$ is also a solution to $\mathbf{x}'(t) = A\mathbf{x}(t)$.
- The solution set of $\mathbf{x}'(t) = A\mathbf{x}(t)$ is an n-dimensional vector space, so if $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are n linearly independent solutions to $\mathbf{x}'(t) = A\mathbf{x}(t)$, then any solution to $\mathbf{x}'(t) = A\mathbf{x}(t)$ is a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

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Example

Let us find the general solution to the system $\mathbf{x}'(t) = A\mathbf{x}(t)$ where $A = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$.

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Attractors, repellers and saddle points

Let $\mathbf{x}'(t) = A\mathbf{x}(t)$ be a system of first order differential equations.

- The origin is called an *attractor* (or a *sink*) if every solution of $\mathbf{x}'(t) = A\mathbf{x}(t)$ converges to $\mathbf{0}$ as $t \to \infty$.
- The origin is called a *repeller* (or a *source*) if every solution of $\mathbf{x}'(t) = A\mathbf{x}(t)$ converges to $\mathbf{0}$ as $t \to -\infty$.
- The origin is called a *saddle point* if there is a solution $\mathbf{x}$ of $\mathbf{x}'(t) = A\mathbf{x}(t)$ that converges to $\mathbf{0}$ as $t \to \infty$ and a solution $\mathbf{y}$ of $\mathbf{x}'(t) = A\mathbf{x}(t)$ that converges to $\mathbf{0}$ as $t \to -\infty$.

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Example

Let us find a solution $\mathbf{x}$ to the system $\mathbf{x}'(t) = A\mathbf{x}(t)$ where $A = \begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix}$, that satisfies $\mathbf{x}(0) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$.

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Decoupling a dynamical system

- Consider a system $\mathbf{x}'(t) = A\mathbf{x}(t)$ of first order differential equations where A is an $n \times n$ matrix with n linearly independent eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$.
- Let $P = [\mathbf{v}_1 \ \ldots \mathbf{v}_n]$ and let D be the diagonal matrix with entries $\lambda_1, \ldots, \lambda_n$. Then $A = PDP^{-1}$.
- If we let $\mathbf{y}(t) = P^{-1}\mathbf{x}(t)$, then $\mathbf{x}'(t) = A\mathbf{x}(t)$ is equivalent to $\mathbf{y}'(t) = D\mathbf{y}(t)$.
- It follows that the general solution to $\mathbf{x}'(t) = A\mathbf{x}(t)$ is $\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + \cdots + c_n\mathbf{v}_n e^{\lambda_n t}$.

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Example

Let us find the general solution to the system $\mathbf{x}'(t) = A\mathbf{x}(t)$ where $A = \begin{bmatrix} 4 & -2 \\ 1 & 6 \end{bmatrix}$.

Complex eigenvalues

Let A be a real 2×2 matrix with a complex eigenvalue λ and corresponding eigenvector $\mathbf{v}$. Let

$$\mathbf{y}_1(t) = \big((\mathrm{Re}(\mathbf{v}))\cos(\mathrm{Im}(\lambda)t) - (\mathrm{Im}(\mathbf{v}))\sin(\mathrm{Im}(\lambda)t)\big)e^{\mathrm{Re}(\lambda)t}$$

$$\mathbf{y}_2(t) = \big((\mathrm{Re}(\mathbf{v}))\sin(\mathrm{Im}(\lambda)t) + (\mathrm{Im}(\mathbf{v}))\cos(\mathrm{Im}(\lambda)t)\big)e^{\mathrm{Re}(\lambda)t}$$

Then $\{\mathbf{y}_1, \mathbf{y}_2\}$ is a basis for the solution space of $\mathbf{x}'(t) = A\mathbf{x}(t)$.

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Writing second-order differential equations as systems of first order differential equations

- Let $y'' + ay' + by = 0$ be a homogeneous second-order differential equation with constant coefficients.
- Let $x_1(t) = y(t)$ and $x_2(t) = y'(t)$.
- Then the equation $y'' + ay' + by = 0$ is equivalent to the system

$$x_1' = x_2$$
$$x_2' = -bx_1 - ax_2$$

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Example

Let us find the solution to the initial values problem

$$y'' - y' - 6y = 0,\ y(0) = 3,\ y'(0) = 4.$$

Problem 6 from August 2011

Two tanks contain salt water. Tank 1 holds 200 litres and tank 2 holds 100 litres. Pure water enters tank 1 at a rate of 4 litres per second, salt water flows from tank 1 to tank 2 at the rate of 9 litres per seconds, salt water flows from tank 2 to tank 1 at the rate of 5 litres per second, and salt water flows from tank 2 and out of the system at a rate of 4 litres per second. The tanks are stirred so that the salt in each is evenly distributed.

1. Find a system of differential equations which describes the salt concentrations $y_1(t)$ and $y_2(t)$ of, respectively, tank 1 and tank 2.
2. Determine $y_1(t)$ and $y_2(t)$ given that at the start ($t = 0$), tank 1 contained 100 grammes of salt and tank 2 contained 200 grammes of salt.