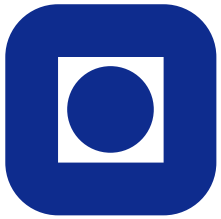

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TMA4115 - Calculus 3

Week 14

Louise Meier Carlsen
Norwegian University of Science and Technology

Spring 2013

Review of last week's lecture

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Review of last week's lecture

Last week we introduce and studied

- the *dimension* of a vector space,

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Review of last week's lecture

Last week we introduce and studied

- the *dimension* of a vector space,
- the *rank* of a matrix,

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Review of last week's lecture

Last week we introduce and studied

- the *dimension* of a vector space,
- the *rank* of a matrix,
- *Markov chains*.

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Today's lecture

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Today we shall introduce and study

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Today's lecture

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- *eigenvectors*, *eigenvalues* and *eigenspaces* of square matrices,

Today's lecture

Today we shall introduce and study

- *eigenvectors*, *eigenvalues* and *eigenspaces* of square matrices,
- the *characteristic polynomial* of a square matrix.

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Eigenvectors and eigenvalues of square matrices

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Eigenvectors and eigenvalues of square matrices

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- An *eigenvalue* of A is a scalar λ such that the equation $A\mathbf{x} = \lambda\mathbf{x}$ has a nontrivial solution.
- If λ is an eigenvalue and $\mathbf{x}$ is an eigenvector such that $A\mathbf{x} = \lambda\mathbf{x}$, then $\mathbf{x}$ is called an *eigenvector (of A) corresponding to* λ and λ is called the *eigenvalue (of A) corresponding to* $\mathbf{x}$.

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Eigenspaces

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Let A be an $n \times n$ matrix and let λ be an eigenvalue of A.

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- The set of of all solution of the equation $(A - \lambda I_n)\mathbf{x} = \mathbf{0}$ (i.e., $\mathrm{Nul}(A - \lambda I_n)$) is called the *eigenspace (of A) corresponding to* λ.

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- Notice that an eigenspace of A is a subspace of $\mathbb{R}^n$.

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- Notice that an eigenspace of A is a subspace of $\mathbb{R}^n$.
- Notice also that $\mathbf{0}$ belongs to any eigenspace, even though it is not an eigenvector.

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Linearly independent eigenvectors

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Linearly independent eigenvectors

Theorem 2

If $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is linearly independent.

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Finding a basis of an eigenspace

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Finding a basis of an eigenspace

Let A be an $n \times n$ matrix and let λ be an eigenvalue of A. The eigenspace of A corresponding to λ is $\text{Nul}(A - \lambda I_n)$.

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Finding a basis of an eigenspace

Let A be an $n \times n$ matrix and let λ be an eigenvalue of A. The eigenspace of A corresponding to λ is $\text{Nul}(A - \lambda I_n)$. So we can find a basis for the eigenspace of A corresponding to λ by:

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2. write the solutions to the equation $(A - \lambda I_n)\mathbf{x} = \mathbf{0}$ (which is equivalent to the equation $A\mathbf{x} = \mathbf{0}$) as a linear combinations of vectors using the free variables as parameters.

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3. Then these vectors form a basis of the eigenspace of A corresponding to λ.

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Example

Let $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$.

Let us determine if 2 is an eigenvalue of A, and let us find a basis for the corresponding eigenspace if it is.

Finding the eigenvalues of a matrix

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- A scalar λ is an eigenvalue of A if and only if $\det(A - \lambda I_n) = 0$.
- The equation $\det(A - \lambda I_n) = 0$ is called the *characteristic equation* of A.

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- If we regard λ as an independent variable, then $\det(A - \lambda I_n)$ is a polynomial of degree n in λ.

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- $\det(A - \lambda I_n)$ is called the *characteristic polynomial* of A.

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- The equation $\det(A - \lambda I_n) = 0$ is called the *characteristic equation* of A.
- If we regard λ as an independent variable, then $\det(A - \lambda I_n)$ is a polynomial of degree n in λ.
- $\det(A - \lambda I_n)$ is called the *characteristic polynomial* of A.
- The eigenvalues of A are the zeros of the characteristic polynomial $\det(A - \lambda I_n)$ of A.

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Example

Let us find the eigenvalues of the matrix $A = \begin{bmatrix} 3 & 0 \\ 4 & -2 \end{bmatrix}$.

Example

In a certain region, 5% of a city's population moves to the surrounding suburbs each year, and 3% of the suburban population moves into the city. In 2000, there were 600,000 residents in the city and 400,000 in the suburbs.
Let us try to find a formula for the number of the people in the city and the number of people in the suburbs for each year.

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Markov chains

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Markov chains

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- A *stochastic matrix* is a square matrix whose columns are probability vectors.
- A *Markov chain* is a sequence $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots$ of probability vectors together with a stochastic matrix P such that $\mathbf{x}_1 = P\mathbf{x}_0$, $\mathbf{x}_2 = P\mathbf{x}_1$, $\mathbf{x}_3 = P\mathbf{x}_2$, $\ldots$.

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- A *steady-state vector* (or an *equilibrium vector*) for a stochastic matrix P is a probability vector $\mathbf{q}$ such that $P\mathbf{q} = \mathbf{q}$.

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- A *steady-state vector* (or an *equilibrium vector*) for a stochastic matrix P is a probability vector $\mathbf{q}$ such that $P\mathbf{q} = \mathbf{q}$.
- So a steady-state vector for a stochastic matrix P is a probability vector which is an eigenvector of P corresponding to the eigenvalue 1.

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Problem 7 from the exam from August 2010

Let $A = \begin{bmatrix} 2 & 0 & 2 \\ 1 & -2 & -1 \\ -1 & 6 & 5 \end{bmatrix}$.

1. Solve the equation $A\mathbf{x} = \mathbf{0}$.
2. Find the eigenvalues and eigenvectors of A.

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Problem 6 from August 2012

For which numbers a does $\mathbb{R}^2$ have a basis of eigenvectors of the matrix $\begin{bmatrix} 0 & a \\ 1 & 0 \end{bmatrix}$?

Monday's lecture

Tomorrow we shall look at

- *diagonal* matrices,
- how to *diagonalizable* a matrix.

Section 5.3 in "Linear Algebras and Its Applications" (pages 281—288).

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