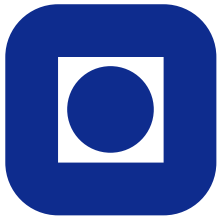

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TMA4115 - Calculus 3

Week 11

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Review of last week

Last week we introduced and studied vector spaces, subspaces, null spaces, column spaces, bases of vector spaces and coordinate systems.

- Vector spaces and subspaces (Section 4.1)
- Null spaces, column spaces, and linear transformations (Section 4.2)
- Linearly independents sets; bases (Section 4.3)
- Coordinate systems (Section 4.4)

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Plan for this week

This week we shall study the dimension of a vector space and the rank of a matrix, we shall look at applications to so-called Markov chains and begin to look at eigenvalues and eigenvectors.

- The dimension of a vector space (Section 4.5)
- Rank (Section 4.6)
- Applications to Markov chains (Section 4.9)
- Eigenvalues and Eigenvectors (Section 5.1)

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The dimension of a vector space

Theorem 9

If a vector space V has a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, then any set in V containing more than n vectors must be linearly dependent.

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The dimension of a vector space

Theorem 10

If a vector space has a basic of n vectors, then every basis of V must consist of exactly n vectors.

The dimension of a vector space

- If V is spanned by a finite set, then V is said to be *finite-dimensional*.
- If V is a finite dimensional vector space, then the *dimension* of V, written as $\dim(V)$, is the number of vectors in a basis for V.
- The dimension of the zero vector space $\{\mathbf{0}\}$ is defined to be zero.
- If V is not spanned by a finite set, then V is said to be *infinite-dimensional*.

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The dimension of some vector spaces

- Since $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is a basis for $\mathbb{R}^n$, the vector space $\mathbb{R}^n$ is finite-dimensional and $\dim(\mathbb{R}^n) = n$.
- Since $\{1, t, \ldots, t^n\}$ is a basis for $\mathbb{P}_n$, the vector space $\mathbb{P}_n$ is finite-dimensional and $\dim(\mathbb{P}_n) = n + 1$.
- The vector space $\mathbb{P}$ is infinite-dimensional.

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Subspaces of a finite-dimensional space

Theorem 11

Let H be a subspace of a finite-dimensional vector space V. Then H is finite-dimensional and $\dim(H) \leq \dim(V)$.
In addition, if $H \neq \{\mathbf{0}\}$, then any linearly independent set in H can be expanded to a basic for H.

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Example

Let us find a basis for the subspace

$$H = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : a, b, c, d \text{ in } \mathbb{R} \text{ and } a + b + c = 0 \text{ and } 2a - d = 0 \right\}$$

of $\mathbb{R}^4$ and $\dim(H)$.

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Problem 4 from the exam from December 2008

Assume that the vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly independent. Show that the vectors $\mathbf{w}_1 = \mathbf{v}_1 + \mathbf{v}_2$, $\mathbf{w}_2 = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$, $\mathbf{w}_3 = \mathbf{v}_1 - \mathbf{v}_2$ are linearly independent.

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The row space of a matrix

- Let A be an $m \times n$ matrix.
- Each row of A can be identified with a vector in $\mathbb{R}^n$.
- The set of all linear combination of the row vectors is called the *row space* of A and is denoted by $\text{Row}(A)$.
- $\text{Row}(A) = \text{Col}(A^T)$.

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Finding a basis for the row space of a matrix

Theorem 13

Let A and B be matrices.

1. If A and B are row equivalent, then $\text{Row}(A) = \text{Row}(B)$.
2. If B is in echelon form, then the nonzero rows of B form a basis for $\text{Row}(B)$.

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Example

Let us find a basis for the row space of the matrix

$$A = \begin{bmatrix} 1 & 4 & -1 & -1 \\ -1 & -4 & 2 & 0 \\ 2 & 8 & -2 & -2 \end{bmatrix}.$$

The rank of a matrix

Let A be a matrix. Then the *rank* of A, written rank(A), is the dimension of the column space Col(A) of A.

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The rank theorem

Theorem 14

Let A be an $m \times n$ matrix. Then the following numbers are equal:

- $\text{rank}(A)$
- $\dim(\text{Col}(A))$
- $\dim(\text{Row}(A))$
- $n - \dim(\text{Nul}(A))$
- The number of pivot positions in A.

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Problem 4 from the exam from August 2011

Let $A = \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ -2 & 3 & -3 & -3 & -4 \\ 4 & -6 & 9 & 5 & 9 \\ -2 & 3 & 3 & -4 & 1 \end{bmatrix}$.

1. Find a basis for the null space, Nul(A), and a basis for the row space, Row(A).
2. Find a basis for the column space, Col(A). What is rank(A).

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Example

If A is 3×4 matrix with a 3-dimensional null space, what is rank(A)?

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Example

Let A be a nonzero 3×3 matrix. Suppose that

$$A \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} = A \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \mathbf{0}.$$

What is $\dim(\text{Row}(A))$?

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Example

Suppose we have a homogeneous linear system consisting of 25 equations in 28 variables. Can we be certain that there the system has 3 linearly independent solutions.

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The invertible matrix theorem

Theorem

Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is invertible.

1. The columns of A form a basis of $\mathbb{R}^n$.
2. $\text{Col}(A) = \mathbb{R}^n$.
3. $\dim(Col(A)) = n$.
4. $\text{rank}(A) = n$.
5. $\text{Nul}(A) = \{\mathbf{0}\}$.
6. $\dim(\text{Nul}(A)) = 0$.
7. $\dim(\text{Row}(A)) = n$.
8. The rows of A form a basis of $\mathbb{R}^n$.
9. $\text{Row}(A) = \mathbb{R}^n$.

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