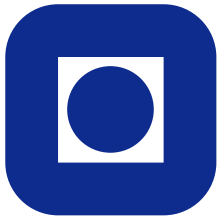

NTNU
Norwegian University of
Science and Technology

TMA4115 - Calculus 3

Week 11

Louise Meier Carlsen
Norwegian University of Science and Technology

Spring 2013

Review of last section

Wednesday we introduced and studied the *determinant* of a square matrix.

- Introduction to determinants (Section 3.1)
- Properties of determinants (Section 3.2)
- Cramer's rule, volume, and linear transformations (Section 3.3)

Plan for this week

This week we shall introduce and study vector spaces, subspaces, null spaces, column spaces and bases of vector spaces.

- Vector spaces and subspaces (Section 4.1)
- Null spaces, column spaces, and linear transformations (Section 4.2)
- Linearly independents sets; bases (Section 4.3)
- Coordinate systems (Section 4.4)

NTNU
Norwegian University of
Science and Technology

Vector spaces

Definition

A *vector space* is a nonempty set V of objects, called vectors, on which are defined two operations, called addition and multiplication by scalars (real numbers), subject to the ten axioms listed below. The axioms must hold for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and for all scalars c and d.

1. The *sum* of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u}+\mathbf{v}$, is in V.
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.
3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.
4. There is a *zero* vector $\mathbf{0}$ in V such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$.

NTNU
Norwegian University of
Science and Technology

1. The *sum* of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u}+\mathbf{v}$, is in V.
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.
3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.
4. There is a *zero* vector $\mathbf{0}$ in V such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$.
5. For each $\mathbf{u}$ in V, there is a vector $-\mathbf{u}$ in V such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.
6. The *scalar multiple* of $\mathbf{u}$ by c, denoted by $c\mathbf{u}$, is in V.
7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$.
8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$.
9. $c(d\mathbf{u})=(cd)\mathbf{u}$.
10. $1\mathbf{u}=\mathbf{u}$.

Examples of vector spaces

1. Let n be a positive integer. Then $\mathbb{R}^n$ is a vector space.
2. Let m and n be positive integers. Then the set $M_{m\times n}$ of all $m \times n$ matrices is a vector space.
3. Let $\mathbb{S}$ be the set of all doubly infinite sequences of numbers: $\{y_k\}_{k\in\mathbb{Z}}$. If $\{z_k\}_{k\in\mathbb{Z}}$ is another element of $\mathbb{S}$, then the sum $\{y_k\}_{k\in\mathbb{Z}} + \{z_k\}_{k\in\mathbb{Z}}$ is sequence $\{y_k + z_k\}_{k\in\mathbb{Z}}$, and if c is a scalar, then $c\{y_k\}_{k\in\mathbb{Z}}$ is the sequence $\{cy_k\}_{k\in\mathbb{Z}}$. Then $\mathbb{S}$ is a vector space.

NTNU
Norwegian University of
Science and Technology

Examples of vector spaces

4. Arrows in a fixed plane, starting at one fixed point. This is used in physics to describe forces or velocities.
   v + w is defined as the diagonal in the parallelogram spanned by v and w. This arrow also has the same starting point.
   rv is defined to be the arrow with the same direction of v, but is dilated or shrunk by multiplying its length by r. When r is negative the arrow is defined to be pointing in the opposite direction instead.

Subspaces

Definition

A *subspace* of a vector space V is a subset H of V that has 3 properties:

1. The zero vector of V is in H.
2. If $\mathbf{u}$ and $\mathbf{v}$ are in H, then $\mathbf{u} + \mathbf{v}$ is in H.
3. If $\mathbf{u}$ is in H and c is a scalar, then $c\mathbf{u}$ is in H.

Every subspace is a vector space. Conversely, every vector space is a subspace (of itself and possibly of other larger spaces).

NTNU
Norwegian University of
Science and Technology

Examples of subspaces

Let V be the space of all real-valued functions defined on $\mathbb{R}$.

1. Let $\mathbb{P}$ be the set of all polynomials with real coefficients. Then $\mathbb{P}$ is a subspace of V.
2. For each $n \geq 0$, let $\mathbb{P}_n$ be the set of polynomials with real coefficients of degree at most n. Then $\mathbb{P}_n$ is a subspace of $\mathbb{P}$ (and of V).
3. Let $C(\mathbb{R})$ be the set of all real-valued continuous functions defined on $\mathbb{R}$. Then $C(\mathbb{R})$ is a subspace of V.
4. Let $Diff(\mathbb{R})$ be the set of all real-valued differentiable functions defined on $\mathbb{R}$. Then Let $Diff(\mathbb{R})$ is a subspace of V.

NTNU
Norwegian University of
Science and Technology

The zero subspace

Let V be a vector space, and let $\mathbf{0}$ be the zero vector of V. The set consisting of only $\mathbf{0}$ is a subspace of V. This subspace is called the *zero subspace* and is written as $\{\mathbf{0}\}$.

NTNU
Norwegian University of
Science and Technology

Subspaces spanned by sets

Let V be a vector space.

1. If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are vectors in V and $c_1, c_2, \ldots, c_n$ are scalars, then the vector $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$ is called a *linear combination* of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.
2. We let $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ denote the set of all possible linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.
3. $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a subspace of V.
4. $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is called the *subspace spanned* (or *generated*) by $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.
5. Given any subspace H of V, a *spanning* (or *generating*) set for H is a set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ in H such that $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$.

NTNU
Norwegian University of
Science and Technology

Example 10

Let V be a vector space and let $\mathbf{v}_1$ and $\mathbf{v}_2$ be vectors in V, then $Span\{\mathbf{v}_1, \mathbf{v}_2\}$ is a subspace of V.

Example

Let H be the set of all vectors of the form $\begin{bmatrix} 2a-3b \\ 0 \\ -a+b \end{bmatrix}$

where a and b represent arbitrary real numbers.
Let us determine if H is a subspace of $\mathbb{R}^3$.

NTNU
Norwegian University of
Science and Technology

Example

Let H be the set of all vectors of the form $\begin{bmatrix} 4a+3b \\ -1 \\ -a+2b \end{bmatrix}$

where a and b represent arbitrary real numbers.
Let us determine if H is a subspace of $\mathbb{R}^3$.

NTNU
Norwegian University of
Science and Technology

The null space of a matrix

Let A be an $m \times n$ matrix.

1. The *null space* of A, written as $\text{Nul}(A)$, is the set of all solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.
2. $\text{Nul}(A) = \{\mathbf{x} : \mathbf{x} \text{ is in } \mathbb{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\}$.
3. $\text{Nul}(A)$ is a subspace of $\mathbb{R}^n$.

NTNU
Norwegian University of
Science and Technology

Example

Let $A = \begin{bmatrix} 3 & 4 & 0 \\ -1 & 2 & 2 \end{bmatrix}$ and let $\mathbf{x} = \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix}$.

Let us determine if $\mathbf{x}$ is in Nul(A).

NTNU
Norwegian University of
Science and Technology

Example

Let $A = \begin{bmatrix} 3 & 4 & 0 \\ -1 & 2 & 2 \end{bmatrix}$.

Let us find an explicit description of Nul(A) by listing vectors that span Nul(A).

Example

Let H be the set of vectors $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ which satisfy that

$2a + 2b = c$ and $a + b + c = 3d$.

Let us determine if H is a vector space.

The column space of a matrix

Let A be an $m \times n$ matrix.

1. The *column space* of A, written as $\text{Col}(A)$, is the set of all linear combinations of the columns of A.
2. If $A = [\mathbf{a}_1\ \mathbf{a}_2 \ldots \mathbf{a}_n]$, then $\text{Col}(A) = \text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$.
3. $\text{Col}(A) = \{\mathbf{b} : \mathbf{b} = A\mathbf{x} \text{ for some } \mathbf{x} \text{ in } \mathbb{R}^n\}$.
4. $\text{Col}(A)$ is a subspace of $\mathbb{R}^m$.

NTNU
Norwegian University of
Science and Technology

Example

Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 0 & 2 \\ 0 & 3 & -3 \end{bmatrix}$.

Then $\text{Col}(A) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix} \right\}$.

Let $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Let us determine if $\mathbf{b}$ is in $\text{Col}(A)$.

NTNU
Norwegian University of
Science and Technology

Example

Let H be the set of vectors of the form $\begin{bmatrix} 2r - s \\ r \\ -3s \end{bmatrix}$

where r and s represent arbitrary real numbers.
Let us determine if H is a vector space.

NTNU
Norwegian University of
Science and Technology

Linear transformations

Definition

A *linear transformation* T from a vector space V into a vector space W is a rule that assigns to each vector $\mathbf{x}$ in V a unique vector $T(\mathbf{x})$ in W such that

1. $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in V,
2. $T(c\mathbf{v}) = cT(\mathbf{v})$ for all $\mathbf{u}$ in V all scalars c.

NTNU
Norwegian University of
Science and Technology

The kernel and range of a linear transformation

Let V and W be vector spaces and let T be a linear transformation from V to W.

1. The *kernel* (or *null space*) of T is the set $\{\mathbf{u} : \mathbf{u} \text{ is in } V \text{ and } T(\mathbf{v}) = \mathbf{0}\}$ of all $\mathbf{u}$ in V for which $T(\mathbf{u}) = \mathbf{0}$.
2. The *range* of T is the set $\{\mathbf{w} : \mathbf{w} = T(\mathbf{v}) \text{ for some } \mathbf{v} \text{ in } V\}$ of all vectors in W of the form $T(\mathbf{v})$ for some $\mathbf{v}$ in V.
3. The kernel of T is a subspace of V.
4. The range of T is a subspace of W.
5. If $V = \mathbb{R}^n$, $W = \mathbb{R}^m$, and A is the standard matrix of T, then the kernel of T is the null space of A, and the range of T is the column space of A.

NTNU
Norwegian University of Science and Technology

Example

Define $T : \mathbb{P} \to \mathbb{R}$ by $T(\mathbf{p}) = \mathbf{p}(1)$. Let us check that T is a linear transformation and describe the kernel and range of T.

NTNU
Norwegian University of
Science and Technology