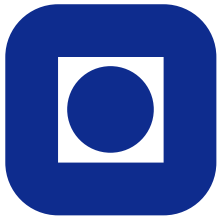

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TMA4110 - Calculus 3

Week 11

Louise Meier Carlsen
Norwegian University of Science and Technology

Spring 2013

Plan for this week

This week we shall introduce and study vector spaces, subspaces, null spaces, column spaces and bases of vector spaces.

- Vector spaces and subspaces (Section 4.1)
- Null spaces, column spaces, and linear transformations (Section 4.2)
- Linearly independents sets; bases (Section 4.3)
- Coordinate Systems (Section 4.4)

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Linear independence

Let V be a vector space, and let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be an indexed set of vectors in V.

- $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is said to be *linearly independent* if the vector equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$ has only the trivial solution.
- $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is said to be *linearly dependent* if there exist scalars $c_1, c_2, \ldots, c_n$ not all equal to zero such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$.

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Linear independence in $\mathbb{R}^n$

- Recall that if $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ are vectors in $\mathbb{R}^n$, then the indexed set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is linearly independent if and only if the matrix $[\mathbf{v}_1\ \mathbf{v}_2 \ldots \mathbf{v}_p]$ has a pivot position in every column.
- There is no similar method to determine if a indexed set of vectors in an arbitrary vector space is linearly independent.

Example

Let $\mathbf{p}_1$, $\mathbf{p}_2$ and $\mathbf{p}_3$ be the elements of $\mathbb{P}$ (the vector space of polynomials with real coefficients) given by $\mathbf{p}_1(t) = 1$, $\mathbf{p}_2(t) = t - 1$ and $\mathbf{p}_3(t) = t^2 + t$.
Let us show that $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is linearly independent.

Linear dependence

Theorem 4

An indexed set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ of two or more vectors in a vector set V is linearly dependent if and only if $\mathbf{v}_1 = \mathbf{0}$ or some $\mathbf{v}_j$, with $j > 1$, is a linear combination of the preceding vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{j-1}$.

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Example

Let $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ and $\mathbf{p}_4$ be the elements of $\mathbb{P}$ (the vector space of polynomials with real coefficients) given by $\mathbf{p}_1(t) = 1$, $\mathbf{p}_2(t) = t$, $\mathbf{p}_3(t) = t^2 - 2$ and $\mathbf{p}_4(t) = t^2 + 2t - 3$.
Then $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4\}$ is linearly dependent because $\mathbf{p}_4 = \mathbf{p}_3 + 2\mathbf{p}_2 - \mathbf{p}_1$.

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Bases

Definition

Let H be a subspace of a vector space V. An indexed set $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ of vectors in H is a *basis* for H if

1. $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ is linearly independent,
2. $H = \text{Span}\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$.

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The standard basis for $\mathbb{R}^n$

- Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ be the columns of I_n.
- That is $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$, $\ldots$, $\mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$.
- Since I_n is invertible, it follows from the inverse matrix theorem that $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ spans $\mathbb{R}^n$ and is linearly independent.
- So $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is a basis for $\mathbb{R}^n$.
- $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is called the *standard basis* for $\mathbb{R}^n$.

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Bases of $\mathbb{R}^n$

- Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be vectors in $\mathbb{R}^n$.
- If the matrix $[\mathbf{v}_1\ \mathbf{v}_2 \ldots \mathbf{v}_n]$ is invertible, then it follows from the inverse matrix theorem that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ spans $\mathbb{R}^n$ and is linearly independent.
- So $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for $\mathbb{R}^n$ if the matrix $[\mathbf{v}_1\ \mathbf{v}_2 \ldots \mathbf{v}_n]$ is invertible.

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Example

Let us show that the set $\{1, t, t^2, \ldots, t^n\}$ is a basis for $\mathbb{P}_n$.

Bases for the solution set of homogeneous linear second-order differential equations

- Let $y'' + p(t)y' + q(t)y = 0$ be a homogenous second-order linear differential equation.
- Recall that the set of solutions to $y'' + p(t)y' + q(t)y = 0$ is the kernel of the linear transformation that maps a real-valued twice continuously differentiable function f to $f'' + p(t)f' + q(t)f$.
- Recall that if y_1 and y_2 are two linearly independent solutions to $y'' + p(t)y' + q(t)y = 0$, then $\{y_1, y_2\}$ is called a fundamental set of solutions.

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Bases for the solution set of homogeneous linear second-order differential equations

- Recall also that if $\{y_1, y_2\}$ is a fundamental set of solutions, then every solution of $y'' + p(t)y' + q(t)y = 0$ is a linear combination of y_1 and y_2.
- So a fundamental set of solutions is a basis for the solutions set of $y'' + p(t)y' + q(t)y = 0$.

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Example

Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix}$, and let $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

Let us find a basis for H.

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The spanning set theorem

Theorem 5

Let V be a vector space, let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be an indexed set of vectors in V, and let $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$.

1. If one of the vectors in $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, say $\mathbf{v}_k$, is a linear combination of the remaining vectors in $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, then the set formed from $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ by removing $\mathbf{v}_k$ still spans H.
2. If $H \neq \{\mathbf{0}\}$, then some subset of $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for H.

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Example

Let $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ and $\mathbf{p}_4$ be the elements of $\mathbb{P}$ given by $\mathbf{p}_1(t) = t$, $\mathbf{p}_2(t) = t^2 + t$, $\mathbf{p}_3(t) = -t^2 + 2t$ and $\mathbf{p}_4(t) = -t^2 + 3t$, and let $H = \text{Span}\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4\}$.
Let us find a basis for H.

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Example

Let us find a basis for Col(B) where

$$B = [\mathbf{b}_1\ \mathbf{b}_2\ \mathbf{b}_3\ \mathbf{b}_4\ \mathbf{b}_5] = \begin{bmatrix} 1 & 2 & 0 & -3 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

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Example

Let us find a basis for Col(A) where

$$A = [\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3\ \mathbf{a}_4] = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 1 & -1 \\ 0 & 0 & -2 & 6 \end{bmatrix}.$$

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Bases for the column space of a matrix

Theorem 6

Let A be an $m \times n$ matrix. Then the pivot columns of A form a basis for $\mathrm{Col}(A)$.

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Example

Let us find a basis for Nul(A) where $A = \begin{bmatrix} 1 & -2 & 0 & 3 \\ -2 & 4 & 1 & -9 \\ -1 & 2 & 1 & -6 \end{bmatrix}$.

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Finding a basis for Nul(A)

1. Row reduce A to its reduced echelon form.
2. Write the solutions to the equation $A\mathbf{x} = \mathbf{0}$ as a linear combinations of vectors using the free variables as parameters.
3. These vectors then form a basis for Nul(A).

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The Unique Representation Theorem

Theorem 7

Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \cdots, b_n\}$ be a basis for vector space V. Then for each $\mathbf{x}$ in V, there exists a unique set of scalars $c_1, \cdots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n.$$

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Coordinates

Let $\mathcal{B} = \{\mathbf{b}_1.\mathbf{b}_2, \cdots, \mathbf{b}_n\}$ be a basis for V and let $\mathbf{x}$ be a vector in V.

1. The coordinates of $\mathbf{x}$ relative to the basis V are the weights $c_1, c_2, \ldots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n.$$

2. If $c_1, c_2, \ldots, c_n$ are $\mathcal{B}$-coordinates of $\mathbf{x}$, then the vector in $\mathbb{R}^n$, $[\mathbf{x}]_{\mathcal{B}} = [c_1, c_2, \ldots, \mathbf{c}_n]^T$, is the coordinate vector of $\mathbf{x}$ (relative ti $\mathcal{B}$), or the $\mathcal{B}$-coordinate vector of $\mathbf{x}$.
3. The mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is called the coordinate mapping (determined by $\mathcal{B}$).

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Example

Let $\mathbf{b}_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ be the basis for $\mathbb{R}^2$. Suppose $\mathbf{x} \in \mathbb{R}$ has the coordinate vector $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$. Find $\mathbf{x}$.

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Example

Let $\mathbf{b}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$. Find the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ of $\mathbf{x}$ relative to $\mathcal{B}$.

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Change-of-coordinates matrix

Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_n\}$ be a basis for $\mathbb{R}^n$.

1. The augmented matrix

$$P_{\mathcal{B}} = [\mathbf{b}_1 \mathbf{b}_2 \cdots \mathbf{b}_n]$$

   is called the change-of-coordinates matrix, and changes the $\mathcal{B}$-coordinates of a vector $\mathbf{x}$ into the standard coordinates for $\mathbf{x}$.

2. The vector equation $\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n$ is equivalent to $\mathbf{x} = P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}$.
3. $P_{\mathcal{B}}^{-1}$ is the coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$.

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The Coordinate Mapping

Theorem 8

Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_n\}$ be a basis for the vector space V. Then the coordinate mapping $\mathbf{x} \mapsto [x]_{\mathcal{B}}$ is a one-to-one linear transformation from V onto $\mathbb{R}^n$.

One-to one: This means that if $\mathbf{u}$ is different from $\mathbf{v}$ then $[\mathbf{u}]_{\mathcal{B}}$ is different from $[\mathbf{v}]_{\mathcal{B}}$.
Linearity of the mapping means that for $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ in V and scalars $c_1, c_2, \cdots, c_p$ then
$[c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_p\mathbf{u}_p]_{\mathcal{B}} = [c_1\mathbf{u}_1]_{\mathcal{B}} + [c_2\mathbf{u}_2]_{\mathcal{B}} + \cdots + [c_p\mathbf{u}_p]_{\mathcal{B}}$.

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Example

Determine if the following polynomials are linear independent in $\mathbb{P}_3$: $1 + 2t^2$, $2 + t^2 - t^3$, $t^2 + 4t^3$ and $3 + 2t$.