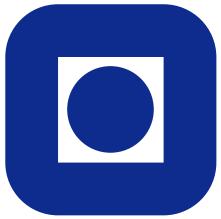

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TMA4115 - Calculus 3

Week 10

Louise Meier Carlsen
Norwegian University of Science and Technology

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Plan for this week

This week we will look at when a matrix has an inverse, an algorithm for finding the inverse, define and study the determinant of a square matrix.

- Me
- The inverse of a matrix (Section 2.2)
- Characterizations of invertible matrices (Section 2.3)
- Introduction to determinants (Section 3.1)
- Properties of determinants (Section 3.2)
- Cramer's Rule, volume and linear transformations (Section 3.3)

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Determinant

- Motivation: Google “determinant matrix wiki”.
- The determinant is a value associated with a square matrix.
- $\det(AB) = \det(A)\det(B)$.
- A square matrix A is invertible if and only if $\det(A) \neq 0$.
- The determinant can be used to give an explicit formula for the inverse of an invertible matrix (this is know as *Cramer's rule*).
- The absolute value of the determinant gives the scale factor by which area or volume is multiplied under the associated linear transformation.

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Determinant

- It is obvious that a 1×1 matrix $[a]$ is invertible if and only if $a \neq 0$.
- Recall that a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if $det(A) = ad - bc \neq 0$.
- If $a_{11} \neq 0$ and $a_{11}a_{22} - a_{12}a_{21} \neq 0$, then the 3×3 matrix $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is invertible if and only if $a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} \neq 0$ (we will see that we don't have to assume that $a_{11} \neq 0$ and $a_{11}a_{22} - a_{12}a_{21} \neq 0$).

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The definition of the determinant

For any square matrix A, let A_{ij} denote the submatrix formed by deleting the ith row and the jth column of A.

Definition

For $n \geq 2$, the *determinant* of an $n \times n$ matrix $A = [a_{ij}]$ is

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(A_{1j}).$$

Example

Let us compute the determinant of $A = \begin{bmatrix} 1 & 2 & -2 \\ 0 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}$.

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Cofactor expansions

When $A = [a_{ij}]$, the (i, j)-cofactor of A is the number $C_{ij} = (-1)^{i+j} \det(A_{ij})$.

Theorem 1

Let $A = [a_{ij}]$ be an $n \times n$ matrix. Then

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

and

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

for any i and any j between 1 and n.

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Example

Let us compute the determinant of $A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$ by using a cofactor expansion across the third column.

Another example $B = \begin{bmatrix} 1 & 2 & -2 & 4 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

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The determinant of a triangular matrix

A *triangular* matrix is a square matrix $A = [a_{ij}]$ for which $a_{ij} = 0$ when $i > j$.

Theorem 2

If A is a triangular matrix, then $\det(A)$ is the product of the entries on the main diagonal of A.

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Properties of determinants

Theorem 3

Let A be a square matrix.

1. If a multiple of one row of A is added to another row to produce a matrix B, then $\det(B) = \det(A)$.
2. If two rows of A are interchanged to produce B, then $\det(B) = -\det(A)$.
3. If one row of A is multiplied by k to produce B, then $\det(B) = k\det(A)$.

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Example

Let us look at three examples and determine the determinant:

$$A = \begin{bmatrix} -2 & 1 & 0 \\ 4 & -1 & 1 \\ 0 & 0 & 3 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 4 & -7 \\ 0 & 0 & 3 \end{bmatrix},$$

$$C = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}.$$

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Properties of determinants

- Suppose an $n \times n$ matrix A has been reduced to an echelon form U by row replacements and row interchanges.
- If there are r interchanges, then $\det(A) = (-1)^r \det(U)$.
- Since U is in echelon form, it is triangular, so $\det(U)$ is the product of the diagonal entries $u_{11}, u_{22}, \ldots, u_{nn}$.
- If A is invertible, the entries $u_{11}, u_{22}, \ldots, u_{nn}$ are all pivots. Otherwise, at least one u_{ii} is zero.
- Thus,

$$\det(A) = \begin{cases} (-1)^r u_{11} u_{22} \ldots u_{nn} & \text{when } A \text{ is invertible} \\ 0 & \text{when } A \text{ is not invertible} \end{cases}$$

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Properties of determinants

Theorem 4

A square matrix A is invertible if and only if $\det(A) \neq 0$.

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Column operations

Theorem 5

If A is a square matrix, then $\det(A^T) = \det(A)$.

Elementary matrices and determinants

If A is an $n \times n$ matrix and E is an $n \times n$ elementary matrix, then $\det(EA) = \det(E)\det(A)$ where

$$\det(E) = \begin{cases} 1 & \text{if } E \text{ is a row replacement matrix} \\ -1 & \text{if } E \text{ is a row interchange matrix} \\ k & \text{if } E \text{ is a scale by } k \text{ matrix} \end{cases}$$

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Multiplicative property

Theorem 6

If A and B are $n \times n$ matrices, then $\det(AB) = \det(A)\det(B)$.

It follows from the theorem that if A and B are $n \times n$ matrices, then $\det(AB) = det(A)\det(B) = \det(B)\det(A) = \det(BA)$, even if $AB \neq BA$.

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Warnings

Let A and B be $n \times n$ matrices and let k be a scalar.

- In general, $\det(A + B) \neq \det(A) + \det(B)$.
- In general, $\det(kA) \neq k \det(A)$.

In fact, $\det(kA) = k^n \det(A)$.

Cramer's rule

For any $n \times n$ matrix A and any $\mathbf{b}$ in $\mathbb{R}^n$, let $A_i(\mathbf{b})$ be the matrix obtained from A by replacing column i by the vector $\mathbf{b}$.

Theorem 7

Let A be an invertible $n \times n$ matrix. For any $\mathbf{b}$ in $\mathbb{R}^n$, the unique solution $\mathbf{x}$ of the equation $A\mathbf{x} = \mathbf{b}$ has entries given by

$$x_i = \frac{\det(A_i(\mathbf{b}))}{\det(A)} \text{ for } i = 1, 2, \ldots, n.$$

Example

Let us use Cramer's rule to solve the linear system

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 4 \\ -x_1 + 2x_3 &= 2 \\ 3x_1 + x_2 + 3x_3 &= -2 \end{aligned}$$

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Example

Let us find the values of the parameter s for which the system

$$2sx_1 + x_2 = 1$$
$$3sx_1 + 6sx_2 = 2$$

has a unique solution, and then find this solution.

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An inverse formula

When A is an $n \times n$ matrix, then $\text{adj}(A)$ is the $n \times n$ matrix whose (i, j)-entry is C_{ji}.

Theorem 8

Let A be an invertible $n \times n$ matrix. Then

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A).$$

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Example

Let us find the inverse of the matrix $A = \begin{bmatrix} 3 & 6 & 7 \\ 0 & 2 & 1 \\ 2 & 3 & 4 \end{bmatrix}$

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Areas and linear transformations

If T is a transformation and S is a set in the domain of T, then we let $T(S)$ denote the set of images of points in S.

Theorem 10

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation determined by a 2×2 matrix A. If S is a region in $\mathbb{R}^2$ with finite area, then area of $T(S) = |\det(A)|$area of S.
Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation determined by a 3×3 matrix A. If S is a region in $\mathbb{R}^2$ with finite volume, then volume of $T(S) = |\det(A)|$volume of S.

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Example

Let us find the area of the parallelogram whose vertices are $(-2, 0)$, $(-3, 3)$, $(2, -5)$ and $(1, -2)$.

Example

Let a and b be positive numbers. Let us find the area of the region E bounded by the ellipse whose equation is

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1.$$

Problem 5 from the exam from August 2011

Let $A = \begin{bmatrix} 1 & 2 & a \\ 3 & 7 & b \\ 2 & 9 & c \end{bmatrix}$.

1. Decide for which values of a, b and c, the matrix A is invertible.
2. Find values of a, b and c for which A^{-1} is an integer matrix.

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