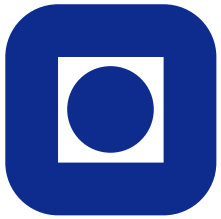

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TMA4115 - Calculus 3

Week 10

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Plan for this week

This week we will look at when a matrix has an inverse, an algorithm for finding the inverse, define and study the determinant of a square matrix.

- Me
- The inverse of a matrix (Section 2.2)
- Characterizations of invertible matrices (Section 2.3)
- Introduction to determinants (Section 3.1)
- Properties of determinants (Section 3.2)
- Cramer's Rule, volume and linear transformations (Section 3.3)

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The inverse of a matrix

1. An $n \times n$ matrix A is said to be *invertible* if there exists an $n \times n$ matrix C such that $AC = CA = I_n$.
2. There can at most be one $n \times n$ matrix C such that $AC = CA = I_n$. If such a C exists, then it is called the *inverse* of A and is denoted by A^{-1}.
3. If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation and A is the standard matrix of T, then T is invertible if and only if A is.
4. If T and A are invertible, then A^{-1} is the standard matrix of T^{-1}.
5. A matrix that is not invertible is sometimes called a *singular* matrix, and an invertible matrix is called a *nonsingular* matrix.

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Example

Let $A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$ and $C = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$.

The inverse of an invertible 2×2 matrix

Theorem 4

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad - bc = 0$, then A is not invertible.

The number $ad - bc$ is called the *determinant* of A.

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Using the inverse of a matrix to solve matrix equations

Theorem 5

If A is an invertible $n \times n$ matrix, then for each $\mathbf{b}$ in $\mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

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Example

Let us solve the linear system

$$3x_1 + 4x_2 = 2$$
$$2x_1 + 3x_2 = 5$$

Properties of invertible matrices

Theorem 6

1. If A is an invertible matrix, then A^{-1} is invertible and $(A^{-1})^{-1} = A$.
2. If A and B are invertible $n \times n$ matrices, then so is AB, and $(AB)^{-1} = B^{-1}A^{-1}$.
3. If A is an invertible matrix, then so is A^T, and $(A^T)^{-1} = (A^{-1})^T$.

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Invertible matrices

Theorem 7

An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduces A to I_n also transforms I_n into A^{-1}.

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An algorithm for finding A^{-1}

1. Row reduce the augmented matrix $[A\ I]$.
2. If A is row equivalent to I, then $[A\ I]$ is row equivalent to $[I\ A^{-1}]$.
3. If A is not row equivalent to I, then A is not invertible.

Problem 4 from the exam from December 2010

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$. Show that A is invertible and find A^{-1}.

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The invertible matrix theorem

Theorem 8

Let A be a square $n \times n$ matrix. Then the following statements are equivalent, that is, for a given A, the following statements are either all true or all false.

1. A is invertible.
2. A is row equivalent to I_n.
3. A has n pivot positions.
4. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
5. The columns of A form a linearly independent set.
6. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.
7. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

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1. A is invertible.
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5. The columns of A form a linearly independent set.
6. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.
7. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.
8. The columns of A span $\mathbb{R}^n$.
9. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is onto.
10. There is an $n \times n$ matrix C such that $CA = I_n$.
11. There is an $n \times n$ matrix D such that $AD = I_n$.
12. A^T is invertible.

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Examples

Let us use the invertible matrix theorem to determine which of the following matrices are invertible.

$$\begin{bmatrix} 1 & -7 & 3 \\ 0 & 0 & 0 \\ 2 & 4 & 4 \end{bmatrix} \qquad \begin{bmatrix} 2 & 4 \\ 1 & -2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & -3 \\ -2 & 1 & 8 \\ -3 & -2 & 5 \end{bmatrix}$$

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Problem 7 from the exam from August 2011

Let A and B be square $n \times n$ matrices and assume that AB is invertible. Show that A is also invertible.

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Determinant

- Motivation: Google “determinant matrix wiki”.
- The determinant is a value associated with a square matrix.
- $\det(AB) = \det(A)\det(B)$.
- A square matrix A is invertible if and only if $\det(A) \neq 0$.
- The determinant can be used to give an explicit formula for the inverse of an invertible matrix (this is know as *Cramer's rule*).
- The absolute value of the determinant gives the scale factor by which area or volume is multiplied under the associated linear transformation.

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Determinant

- It is obvious that a 1×1 matrix $[a]$ is invertible if and only if $a \neq 0$.
- Recall that a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if $det(A) = ad - bc \neq 0$.
- If $a_{11} \neq 0$ and $a_{11}a_{22} - a_{12}a_{21} \neq 0$, then the 3×3 matrix $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is invertible if and only if $a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} \neq 0$ (we will see that we don't have to assume that $a_{11} \neq 0$ and $a_{11}a_{22} - a_{12}a_{21} \neq 0$).

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The definition of the determinant

For any square matrix A, let A_{ij} denote the submatrix formed by deleting the ith row and the jth column of A.

Definition

For $n \geq 2$, the *determinant* of an $n \times n$ matrix $A = [a_{ij}]$ is

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(A_{1j}).$$

Example

Let us compute the determinant of $A = \begin{bmatrix} 1 & 2 & -2 \\ 0 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}$.

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Cofactor expansions

When $A = [a_{ij}]$, the (i, j)-cofactor of A is the number $C_{ij} = (-1)^{i+j} \det(A_{ij})$.

Theorem 1

Let $A = [a_{ij}]$ be an $n \times n$ matrix. Then

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

and

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

for any i and any j between 1 and n.

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Example

Let us compute the determinant of $A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$ by using a cofactor expansion across the third column.

Another example $B = \begin{bmatrix} 1 & 2 & -2 & 4 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

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The determinant of a triangular matrix

A *triangular* matrix is a square matrix $A = [a_{ij}]$ for which $a_{ij} = 0$ when $i > j$.

Theorem 2

If A is a triangular matrix, then $\det(A)$ is the product of the entries on the main diagonal of A.

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Properties of determinants

Theorem 3

Let A be a square matrix.

1. If a multiple of one row of A is added to another row to produce a matrix B, then $\det(B) = \det(A)$.
2. If two rows of A are interchanged to produce B, then $\det(B) = -\det(A)$.
3. If one row of A is multiplied by k to produce B, then $\det(B) = k\det(A)$.

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Properties of determinants

- Suppose an $n \times n$ matrix A has been reduced to an echelon form U by row replacements and row interchanges.
- If there are r interchanges, then $\det(A) = (-1)^r \det(U)$.
- Since U is in echelon form, it is triangular, so $\det(U)$ is the product of the diagonal entries $u_{11}, u_{22}, \ldots, u_{nn}$.
- If A is invertible, the entries $u_{11}, u_{22}, \ldots, u_{nn}$ are all pivots. Otherwise, at least one u_{ii} is zero.
- Thus,

$$\det(A) = \begin{cases} (-1)^r u_{11} u_{22} \ldots u_{nn} & \text{when } A \text{ is invertible} \\ 0 & \text{when } A \text{ is not invertible} \end{cases}$$

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Properties of determinants

Theorem 4

A square matrix A is invertible if and only if $\det(A) \neq 0$.

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Column operations

Theorem 5

If A is a square matrix, then $\det(A^T) = \det(A)$.

Elementary matrices and determinants

If A is an $n \times n$ matrix and E is an $n \times n$ elementary matrix, then $\det(EA) = \det(E)\det(A)$ where

$$\det(E) = \begin{cases} 1 & \text{if } E \text{ is a row replacement matrix} \\ -1 & \text{if } E \text{ is a row interchange matrix} \\ k & \text{if } E \text{ is a scale by } k \text{ matrix} \end{cases}$$

Multiplicative property

Theorem 6

If A and B are $n \times n$ matrices, then $\det(AB) = \det(A)\det(B)$.

It follows from the theorem that if A and B are $n \times n$ matrices, then $\det(AB) = det(A)\det(B) = \det(B)\det(A) = \det(BA)$, even if $AB \neq BA$.

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Warnings

Let A and B be $n \times n$ matrices and let k be a scalar.

- In general, $\det(A + B) \neq \det(A) + \det(B)$.
- In general, $\det(kA) \neq k \det(A)$.

In fact, $\det(kA) = k^n \det(A)$.