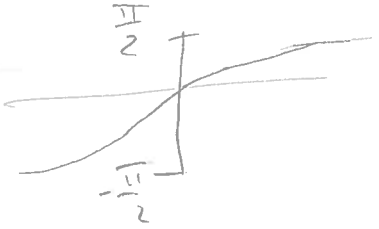
Spring 2012

Prob 1: i) Solve $w^2 = \frac{-1+\sqrt{3}i}{2}$ (Roots of complex numbers)

Let $z = \frac{-1+\sqrt{3}i}{2}$ then $w^2 = z$. Let $r = |z|$ and $\theta = Arg(z)$

then $z = r(\cos\theta + i\cdot\sin\theta)$ and

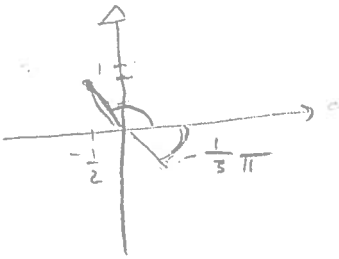
$$w = \pm\sqrt{r}\left(\cos\frac{\theta}{2} + i\cdot\sin\frac{\theta}{2}\right)$$



So $r = |z| = \left|-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$

$$= \sqrt{\frac{1}{4}+\frac{3}{4}} = \sqrt{1} = 1.$$

$\tan(\arg(z)) = \frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = -\sqrt{3}$, so $\arg(z) = \tan^{-1}(-\sqrt{3})$



$= -\tan^{-1}(\sqrt{3}) = -\frac{1}{3}\cdot\pi + \pi$

$= \frac{2}{3}\pi = Arg(z)$, since

between $-\pi < \theta \le \pi$.

$-\frac{1}{3}\pi + \frac{3}{3}\pi = \frac{2}{3}\pi$

$$w = \pm\sqrt{1}\left(\cos\left(\frac{\frac{2}{3}\pi}{2}\right) + i\sin\left(\frac{\frac{2}{3}\pi}{2}\right)\right)$$

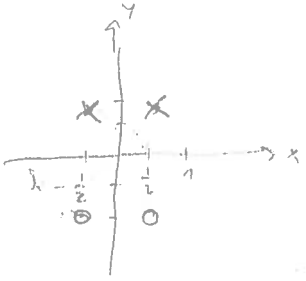
$$= \pm\left(\cos\left(\tfrac{1}{3}\pi\right) + i\sin\left(\tfrac{1}{3}\pi\right)\right) = \pm\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right).$$

ii) Find all solutions of $z^4 + z^2 + 1 = 0$ and draw.

First we notice the polynomial is an quadratic form, thus

$$z^2 = \frac{-1 \pm \sqrt{1^2 - 4\cdot1\cdot1}}{2\cdot1} = \frac{-1\pm\sqrt{3}i}{2}$$

From i) we then get that $\pm\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$ are roots.



for $z^2 = \frac{-1-\sqrt{3}i}{2}$ then $r = \left|-\frac{1}{2} - \frac{\sqrt{3}}{2}\right| = 1$

$\tan(\arg(\frac{-1-\sqrt{3}i}{2})) = \frac{-\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = \sqrt{3}$

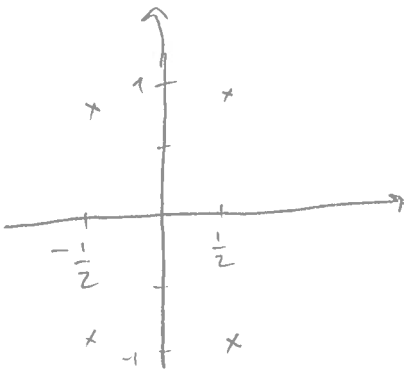
$\tan^{-1}(\sqrt{3}) = \frac{1}{3}\pi - \pi = -\frac{2}{3}\pi$.

so $z = \pm 1\left(\cos\left(\frac{-\frac{2}{3}\pi}{2}\right) + i\sin\left(\frac{-\frac{2}{3}\pi}{2}\right)\right)$

$= \pm\left(\cos\left(-\frac{1}{3}\pi\right) + i\sin\left(-\frac{1}{3}\pi\right)\right)$

$= \pm\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$. ~ o.s.o.

Hence the 4 roots are $\pm\left(\frac{1}{2} \pm i\frac{\sqrt{3}}{2}\right)$.



Section 4.5

Prob 2:

a) Find a particular solution of $y'' - 4y' + y = te^t + t$ (1)

ex 5.23

The forcing term is a sum $f(t) = f_1(t) + f_2(t)$, where $f_1(t) = te^t$ and $f_2(t) = t$. Break up equation:

$y_1'' - 4y_1' + y_1 = te^t$ and find sol. y_1 and y_2

$y_2'' - 4y_2' + y_2 = t$

Then because (1) is linear then $y_1 + y_2$ is a sol. to 1).

i) 1. guess $y_1(t) = ate^t$, $y_1' = ate^t + ae^t$ $y_1'' = ate^t + ae^t + ae^t$

$= ate^t + 2ae^t$

$\Rightarrow y_1'' - 4y_1' + y_1 = ate^t + 2ae^t - 4(ate^t + ae^t) + ate^t$

$= -2ate^t - 2ae^t$ ← too much.

Second guess for $y_1(t)$:

$y_1(t) = at\,e^t + be^t$, then

$y_1'(t) = ate^t + ae^t + be^t = ate^t + (a+b)e^t$

$y_1''(t) = ate^t + ae^t + (a+b)e^t = ate^t + (2a+b)e^t$

By insertion we get

$$y_1'' - 4y_1' + y_1 = ate^t + (2a+b)e^t - 4(ate^t + (a+b)e^t) + ate^t + be^t$$

$$= -2ate^t + (-2a-2b)e^t = te^t$$

Thus

$-2a = 1$ and $(-2a-2b) = 0$

hence $a = -\frac{1}{2}$ and $b = \frac{1}{2}$

Therefore $y_1(t) = -\frac{1}{2}te^t + \frac{1}{2}e^t$

ii) Find guess for $y_2(t)$:

$y_2(t) = c + dt$, then

$y_2'(t) = d$ and $y_2''(t) = 0$

By insertion we get

$$y_2''(t) - 4y_2'(t) + y_2(t) = 0 - 4d + c + dt = t$$

Thus $-4d + c = 0$ and $d = 1$

hence $c = 4$ and $d = 1$

Therefore

$$y_2(t) = 4 + t$$

And the particular solution is

$$y_p(t) = y_1(t) + y_2(t)$$

$$= -\frac{1}{2}te^t + \frac{1}{2}e^t + t + 4 \quad \square$$

b) Find the solution of $y'' - 4y' + y = te^t + t$, where $y'(0) = y(0) = 0$.

i) Find the homogenous solution:

$$y'' - 4y' + y = 0$$

Hence we get the polynomial equation

$$z^2 - 4z + 1 = 0$$

Thus $z = \frac{4 \pm \sqrt{4^2 - 4 \cdot 1 \cdot 1}}{2} = \frac{4 \pm \sqrt{12}}{2} = 2 \pm \sqrt{3}$

Therefor

$$y_h(t) = C_1 e^{(2+\sqrt{3})t} + C_2 e^{(2-\sqrt{3})t}$$

ii) $$y(t) = y_h(t) + y_p(t) = C_1 e^{(2+\sqrt{3})t} + C_2 e^{(2-\sqrt{3})t} - \frac{1}{2}te^t + \frac{1}{2}e^t + t + 4$$

$$y(0) = C_1 + C_2 + \frac{1}{2} + 4 = 0$$

$$y'(t) = C_1(2+\sqrt{3})e^{(2+\sqrt{3})t} + C_2(2-\sqrt{3})e^{(2-\sqrt{3})t}$$

$$-\frac{1}{2}te^t - \cancel{\frac{1}{2}e^t} + \cancel{\frac{1}{2}e^t} + 1$$

$$y'(t) = C_1(2+\sqrt{3}) + C_2(2-\sqrt{3}) + 1 = 0$$

Thus $C_1 = -\frac{9}{2} - C_2$ and

$$\left(-\frac{9}{2} - C_2\right)(2+\sqrt{3}) + C_2(2-\sqrt{3}) = -1$$

$$\Rightarrow -9 - \frac{9\sqrt{3}}{2} + C_2(2 - \sqrt{3} - (2+\sqrt{3})) = -1$$

$$\Rightarrow -C_2(2\sqrt{3}) = 9 + 9\frac{\sqrt{3}}{2} - 1$$

$$\Rightarrow C_2 = -\frac{2\cdot 9 + 9\sqrt{3} - 2}{2}\cdot\frac{1}{2\sqrt{3}} = -\frac{16+9\sqrt{3}}{4\sqrt{3}}$$

so

$$C_1 = -C_2 - \frac{9}{2} = \frac{16+9\sqrt{3}}{4\sqrt{3}} - \frac{9}{2} = \frac{16+9\sqrt{3} - 9\cdot 2\cdot\sqrt{3}}{4\cdot\sqrt{3}}$$

$$= \frac{16-9}{4\cdot\sqrt{3}}$$

Therefore

$$y(t) = \frac{16-9}{4\sqrt{3}} e^{(2+\sqrt{3})t} - \frac{16+9\sqrt{3}}{4\sqrt{3}} e^{(2-\sqrt{3})t} - \frac{1}{2}te^t$$

$$+\frac{1}{2}e^t + t + 4 \quad \square$$

Problem 3:

Let $a \in \mathbb{R}$. Find the general solution of

$$y'' + ay = \cos(x).$$

i) Find homogeneous solution

$$\lambda^2 + a = 0 \Rightarrow \lambda = \pm\sqrt{-a},$$

a) Thus if $a > 0$ we have complex roots and

Let $\sqrt{a} = x + iy$ then $i\sqrt{a}$ but $x=0$ and $y=\sqrt{a}$

$$y_h(t) = A_1 e^{xt}\cos(yt) + A_2 e^{xt}\sin(yt), \text{ where } A_1, A_2 \in \mathbb{R}.$$
$$= A_1\cos(\sqrt{a}t) + A_2\sin(\sqrt{a}t)$$

b) If $a=0$ then $y_h''(t) = 0$ Thus

$$y_h(t) = A_3 + A_4 t$$

c) If $a<0$ then we have two distinct real roots $\sqrt{-a}$ and $-\sqrt{-a}$ and

$$y_u(t) = A_5 e^{\sqrt{-a}t} + A_6 e^{-\sqrt{-a}t}$$

ii) The particular solution:

$$y'' + ay = \cos(t)$$

1. guess:

$$y_p(t) = C_1 \cdot \cos(t) + C_2 \sin(t)$$

then $y_p'(t) = -C_1 \sin(t) + C_2 \cos(t)$

$$y_p''(t) = -C_1 \cos(t) - C_2 \sin(t)$$

and

$$y'' + ay = -C_1 \cos(t) - C_2 \sin(t) + aC_1 \cos(t) + aC_2 \sin(t)$$

$$= (a-1)C_1 \cos(t) + (a-1)C_2 \sin(t) = \cos(t)$$

i) $a=1$ Need new guess

2. guess: $y_p(t) = C_1 t \cos(t) + C_2 t \sin(t)$

then $y_p'(t) = -C_1 t \sin(t) + C_1 \cos(t) + C_2 t \cos(t) + C_2 \sin(t)$

$$= (C_2 - C_1 t)\sin(t) + (C_1 + C_2 t)\cos(t)$$

$$y_p''(t) = (C_2 - C_1 t)\cos(t) - C_1 \sin(t) - (C_1 + C_2 t)\sin(t) + C_2 \cos(t)$$

$$= (2C_2 - C_1 t)\cos(t) - (2C_1 + C_2 t)\sin(t)$$

and

$$y'' + y = (2C_2 - C_1 t)\cos(t) - (2C_1 + C_2 t)\sin(t) + C_1 t \cos(t) + C_2 t \sin(t)$$

$$= 2C_2 \cos(t) - 2C_1 \sin(t) = \cos(t)$$

ok then $C_1 = 0$ and $C_2 = \frac{1}{2}$.

then

$$y(t) = y_p + y_h = \frac{1}{2} t \sin(t) + A_1 \cos(\sqrt{a}\, t) + A_2 \sin(\sqrt{a}\, t) \quad \square$$

ii) If $a \neq 1$ then $C_2 = 0$ and $C_1 = \frac{1}{a-1}$

- and if $a > 0$ we get

$$y(t) = y_p + y_h = \frac{1}{a-1} \cos(t) + A_1 \cos(\sqrt{a}\, t) + A_2 \sin(\sqrt{a}\, t)$$

- if $a = 0$ then

$$y(t) = y_p + y_h = \frac{1}{a-1} \cos(t) + A_3 + A_4 t$$

- if $a < 0$ then

$$y(t) = y_p + y_h = \frac{1}{a-1} \cos(t) + A_5 e^{\sqrt{a}\, i t} + A_6 e^{-\sqrt{a}\, i t} \quad \square$$

Problem 4 Find the least square solution of

$$\begin{aligned} -y+z &= 1,\\ 2y-z &= 0,\\ x+y+3z &= 0,\\ -x+3y+z &= 0. \end{aligned}$$

Let $A=\begin{bmatrix} 0 & -1 & 1 \\ 0 & 2 & -1 \\ 1 & 1 & 3 \\ -1 & 3 & 1 \end{bmatrix}$ $x=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $b=\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ then

$Ax=b$ is equivalent to the above system.

Then the set of least-squares solutions of $Ax=b$ coincides with the nonempty set of solutions of the normal equation $A^TAx=A^Tb$.

$$A^TA=\begin{bmatrix} 0 & 0 & 1 & -1 \\ -1 & 2 & 1 & 3 \\ 1 & -1 & 3 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 & 1 \\ 0 & 2 & -1 \\ 1 & 1 & 3 \\ -1 & 3 & 1 \end{bmatrix}=\begin{bmatrix} 2 & -2 & 2 \\ -2 & 15 & 3 \\ 2 & 3 & 12 \end{bmatrix}$$

$$A^Tb=\begin{bmatrix} 0 & 0 & 1 & -1 \\ -1 & 2 & 1 & 3 \\ 1 & -1 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}=\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 2 & -2 & 2 & 0 \\ -2 & 15 & 3 & -1 \\ 2 & 3 & 12 & 1 \end{array}\right] \xrightarrow[-R_1+R_3]{R_1+R_2} \left[\begin{array}{ccc|c} 2 & -2 & 2 & 0 \\ 0 & 13 & 5 & -1 \\ 0 & 5 & 10 & 1 \end{array}\right] \xrightarrow{\frac{1}{5}\cdot R_3=R_3} \left[\begin{array}{ccc|c} 2 & -2 & 2 & 0 \\ 0 & 13 & 5 & -1 \\ 0 & 1 & 2 & 1/5 \end{array}\right]$$

$$\xrightarrow{-13\cdot R_3+R_2} \left[\begin{array}{ccc|c} 2 & -2 & 2 & 0 \\ 0 & 0 & -21 & -\frac{18}{5} \\ 0 & 1 & 2 & 1/5 \end{array}\right]$$

$\frac{-13}{5}-1$ ÷10

$-\frac{13-5}{5}$

Thus $-21z=-\frac{18}{5} \Rightarrow z=\frac{18}{5\cdot 21}=\frac{6}{5\cdot 7}=\frac{6}{35}$ (÷3.6, ÷7.3)

and $y+2z=1/5 \Rightarrow y=1/5-2\cdot\frac{6}{35}=\frac{7-12}{35}=\frac{-5}{35}=\frac{-1}{7}$

and $x-y+z=0 \Rightarrow x=-\frac{1}{7}-\frac{6}{35}=\frac{-5-6}{35}=\frac{-11}{35}$

so the set of solutions is $\begin{bmatrix} -11/35 \\ -1/7 \\ 6/35 \end{bmatrix}$ □

Problem 5

$T: \mathbb{R}^3 \to \mathbb{R}^4$ defined by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x+y+z \\ -x+3y+z \\ 2x-z \\ y+4z \end{bmatrix}$

a) Find A such that $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = A\begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

$$A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 3 & 1 \\ 2 & 0 & -1 \\ 0 & 1 & 4 \end{bmatrix}$$

b) Find $\dim(\text{Null}(A))$ and a basis for $\text{Col}(A)$. Is T injective? or/and surjective?

i) n – # pivot positions $= \dim(\text{Null}(A))$

$$\begin{bmatrix} 2 & 1 & 1 \\ -1 & 3 & 1 \\ 2 & 0 & -1 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & -3 & -1 \\ 2 & 1 & 1 \\ 2 & 0 & -1 \\ 0 & 1 & 4 \end{bmatrix} \begin{matrix} \xrightarrow{2R_1+R_2} \\ \xrightarrow{-2R_1+R_3} \end{matrix} \begin{bmatrix} 1 & -3 & -1 \\ 0 & 7 & 3 \\ 0 & 6 & 1 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_4} \begin{bmatrix} 1 & -3 & -1 \\ 0 & 1 & 4 \\ 0 & 6 & 1 \\ 0 & 7 & 3 \end{bmatrix}$$

$$\begin{matrix} \xrightarrow{-6 \cdot R_2 + R_3} \\ \xrightarrow{-7 \cdot R_2 + R_4} \end{matrix} \begin{bmatrix} 1 & -3 & -1 \\ 0 & 1 & 4 \\ 0 & 0 & -23 \\ 0 & 0 & -25 \end{bmatrix} \xrightarrow{} \begin{bmatrix} 1 & -3 & -1 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus

$\dim(\text{Null}(A)) = 3 - 3 = 0$ and

$$\left\{ \begin{bmatrix} 2 \\ -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ 4 \end{bmatrix} \right\}$$ is a basis for $\text{Col}(A)$.

ii) Injective if $T(x) = T(y) \Rightarrow x = y$

Yes, since $\dim(\text{Null}(A)) = 0$.

iii) Surjective if for all $v \in \mathbb{R}^4$ there exists $u \in \mathbb{R}^3$ such that $T(u) = v$. But $\dim(\text{Col}(A)) = 3$ and $\dim(\text{Null}(A)) = 0$

Thus $\text{Col}(A) = \mathbb{R}^3 \neq \mathbb{R}^4$ hence not surjective.

Problem 6

A a 4×4 matrix, $B = \begin{bmatrix} 2 & 1 & 4 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $\det(AB) = 4$.

a) What is $\det(A)$?

We know that $\det(AB) = \det(A)\cdot\det(B)$, so

$\det(A) = 4 \cdot \frac{1}{\det(B)}$ if $\det(B) \neq 0$.

$$\det(B) = 1 \cdot \det\begin{bmatrix} 2 & 1 & 4 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = 1 \cdot \det\begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} - 0 + 1 \cdot \det\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= 1 - 4 + 2 - 1 = -2,$$

so $\det(A) = 4 \cdot (-\frac{1}{2}) = -2$.

b) Show that $Ax = 0$ only has the trivial solution.

Since $\det(A) \neq 0$ then A is invertibel and then it follows from the inv. matrix theorem that $Ax = 0$ only has the trivial solution.

Problem 7: Find all eigenvalues of $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & -1 & 4 \end{bmatrix}$

a) $\det(A - \lambda I_3) = 0 \Leftrightarrow \det\begin{bmatrix} 2-\lambda & 0 & 0 \\ 0 & 1-\lambda & 2 \\ 0 & -1 & 4-\lambda \end{bmatrix} = (2-\lambda)\cdot\det\begin{bmatrix} 1-\lambda & 2 \\ -1 & 4-\lambda \end{bmatrix}$

$$= (2-\lambda)\big((1-\lambda)(4-\lambda)+2\big) = (2-\lambda)(\lambda^2 - 4\lambda - \lambda + 4 + 2)$$

$$= (2-\lambda)(\lambda^2 - 5\lambda + 6) = (2-\lambda)(2-\lambda)(3-\lambda)$$

Thus the roots (and also the eigenvalues) are 2, 2 and 3.

b) Find a basis for the eigenspace. Is A diagonalizable?

i) $(A - \lambda I_3)x = 0$

Find nullspace of $(A - \lambda I_3)$.

$\lambda = 2$

$$A - \lambda I_3 = \begin{bmatrix} 2-2 & 0 & 0 \\ 0 & 1-2 & 2 \\ 0 & -1 & 4-2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & -1 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ 2x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} x_3$

So $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\}$ is a basis for the eigenspace corresponding to the eigenvalue 2

$\lambda = 3$

$$A - \lambda I_3 = \begin{bmatrix} 2-3 & 0 & 0 \\ 0 & 1-3 & 2 \\ 0 & -1 & 4-3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & -1 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} x_3$, so $\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for the eigenspace corresponding to the eigenvalue 3

ii) Is A diagonalizable? Since $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a lin. indep. set A is diagonalizable.

(A diag. $\Leftrightarrow$ A has n lin. indep. eigenvectors). § 5.3.

problem 8

$$A = \begin{bmatrix} -2 & -5 \\ 5 & -2 \end{bmatrix}$$

a) Find the complex eigenvalues of A and the corr. eigenvectors in $\mathbb{C}^2$.

i) λ eigenvalue of A $\Leftrightarrow \det(A-\lambda I)=0$

$$\det(A-\lambda I) = \det\begin{pmatrix} -2-\lambda & -5 \\ 5 & -2-\lambda \end{pmatrix} = (-2-\lambda)^2 + 5\cdot 5$$

$$= \lambda^2 + 4\lambda + 4 + 25 = \lambda^2 + 4\lambda + 29 = 0$$

$$\Rightarrow \lambda = \frac{-4 \pm \sqrt{4^2 - 4\cdot 29}}{2} = -2 \pm \sqrt{4-29} = -2 \pm \sqrt{-25}$$

$$= -2 \pm 5i.$$

$\lambda = -2+5i$: Looking for the $\mathrm{Nul}(A-\lambda I_2)$:

$$A-\lambda I_2 = \begin{bmatrix} -2+2-5i & -5 \\ 5 & -2+2-5i \end{bmatrix} = \begin{bmatrix} -5i & -5 \\ 5 & -5i \end{bmatrix} \xrightarrow[=R_1]{i\cdot R_2 + R_1} \begin{bmatrix} 0 & 0 \\ 5 & -5i \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix} \quad \text{so} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ix_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

$\lambda = -2-5i$ Since complex eigenvalues and eigenvectors come in conjugate pairs

$\begin{bmatrix} -i \\ 1 \end{bmatrix}$ is the corr. eigenvector for $-2-5i$.

b) Find the sol of the system of equations

$$y'(t) = Ay(t) \quad \text{such that} \quad y(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Answer on the form

$$y(t) = e^{\lambda t}\begin{bmatrix} a\cos(\omega t) + b\sin(\omega t) \\ c\cos(\omega t) + d\sin(\omega t) \end{bmatrix} \qquad \S 5.7$$

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} -2 & 5 \\ 5 & -2 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

Thus

$$y(t) = C_1 V_1 e^{\lambda_1 t} + C_2 V_2 e^{\lambda_2 t}$$

$$= C_1 \begin{bmatrix} i \\ 1 \end{bmatrix} e^{(-2+5i)t} + C_2 \begin{bmatrix} -i \\ 1 \end{bmatrix} e^{(-2-5i)t}$$

$$y(0) = C_1 \begin{bmatrix} i \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} -i \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad i(C_1 - C_2) = 1 \quad \text{and} \quad C_1 + C_2 = 0$$

$$\Rightarrow C_1 = -C_2 \text{ and } i(-C_2 - C_2) = 1 \Rightarrow C_2 = -\frac{1}{2i}$$

$$\text{and} \quad C_1 = \frac{1}{2i}$$

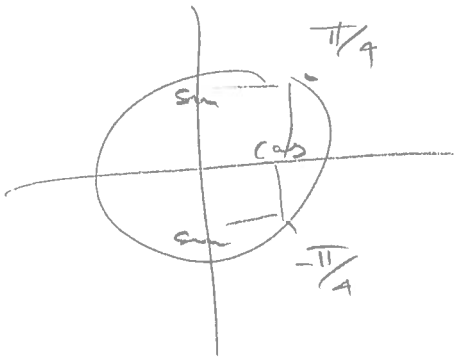
Thus

$$y(t) = \frac{1}{2i}\begin{bmatrix} i \\ 1 \end{bmatrix} e^{(-2+5i)t} = \frac{1}{2i}\begin{bmatrix} -i \\ 1 \end{bmatrix} e^{(-2-5i)t}$$

$$= e^{-2t}\begin{bmatrix} \frac{i}{2i} e^{5it} + \frac{i}{2i} e^{-5it} \\ \frac{1}{2i} e^{5it} - \frac{1}{2i} e^{-5it} \end{bmatrix} =$$

$e^{(a+bi)t} = e^{at}e^{bit} = e^{at}(\cos(bt)+i\sin(bt))$ p. 3.16

so $= e^{-2t}\begin{bmatrix}\frac{1}{2}\left(e^{5it}+e^{-5it}\right)\\ \frac{1}{2i}\left(e^{5it}-e^{-5it}\right)\end{bmatrix} = e^{-2t}\begin{bmatrix}\frac{1}{2}\left(\cos(5t)+i\sin(5t)+\cos(-5t)+i\sin(-5t)\right)\\ \frac{1}{2i}\left(\cos(5t)+i\sin(5t)-\cos(-5t)-i\sin(-5t)\right)\end{bmatrix}$

$= e^{-2t}\begin{bmatrix}\frac{1}{2}\left(2\cos(5t)+i\sin(5t)-i\sin(5t)\right)\\ \frac{1}{2i}\left(\cos(5t)-\cos(5t)+2i\sin(5t)\right)\end{bmatrix}$

$= e^{-2t}\begin{bmatrix}\cos(5t)\\ \sin(5t)\end{bmatrix}$ □