Designers of shock absorbers aim for a damping constant that is just a little larger than critical damping. If the damping constant is too large, the shocks prevent the springs from absorbing bounces, and the ride becomes bumpy. If the damping constant gets below critical damping, the car oscillates. So just a little above critical damping is the right goal. This prevents the oscillations of the underdamped case, and it allows the springs to absorb the bounces. In addition it allows for the damping constant to decrease a little before it becomes underdamped.

EXERCISES

In Exercises 1–6,

(i) use a computer or calculator to plot the graph of the given function, and

(ii) place the solution in the form $y = A\cos(\omega t - \phi)$ and compare the graph of your answer with the plot found in part (i).

1. $y = \cos 2t + \sin 2t$ **2.** $y = \cos t - \sin t$

3. $y = \cos 4t + \sqrt{3}\sin 4t$ **4.** $y = -\sqrt{3}\cos 2t + \sin 2t$

5. $y = 0.2\cos 2.5t - 0.1\sin 2.5t$

6. $y = 0.2\cos 6.3t - 0.5\sin 6.3t$

In Exercises 7–10, place each equation in the form $y = Ae^{-ct}\cos(\omega t - \phi)$. Then, on one plot, place the graphs of $y = Ae^{-ct}\cos(\omega t - \phi)$, $y = Ae^{-ct}$, and $y = -Ae^{-ct}$. For the last two, use a different line style and/or color than for the first.

7. $y = e^{-t/2}(\cos 5t + \sin 5t)$

8. $y = e^{-t/4}(\sqrt{3}\cos 4t - \sin 4t)$

9. $y = e^{-0.1t}(0.2\cos 2t + 0.1\sin 2t)$

10. $y = e^{-0.2t}(\cos 4.2t - 1.2\sin 4.2t)$

11. A 0.2-kg mass is attached to a spring having a spring constant $5\,\text{kg/s}^2$. The system is displaced 0.5 m from its equilibrium position and released from rest. If there is no damping present, find the amplitude, frequency, and phase of the resulting motion. Plot the solution.

12. A 0.1-kg mass is attached to a spring having a spring constant $3.6\,\text{kg/s}^2$. The system is allowed to come to rest. Then the mass is given a sharp tap, imparting an instantaneous downward velocity of 0.4 m/s. If there is no damping present, find the amplitude, frequency, and phase of the resulting motion. Plot the solution.

13. The undamped system

$$\frac{2}{5}x'' + kx = 0, \quad x(0) = 2, \quad x'(0) = v_0$$

is observed to have period $\pi/2$ and amplitude 2. Find k and v_0.

14. Consider the undamped oscillator

$$mx'' + kx = 0, \quad x(0) = x_0, \quad x'(0) = v_0.$$

Show that the amplitude of the resulting motion is $\sqrt{x_0^2 + mv_0^2/k}$.

15. A 2-μF capacitor ($1\,\mu\text{F} = 1 \times 10^{-6}\,\text{F}$) is charged to 20 V and then connected across a 6-μH inductor ($1\,\mu\text{H} = 1 \times 10^{-6}\,\text{H}$), forming an LC circuit.

(a) Find the initial charge on the capacitor.

(b) At the time of connection, the initial current is zero. Assuming no resistance, find the amplitude, frequency, and phase of the current. Plot the graph of the current versus time. (Use equations (4.2) and (4.3) in Section 3.4.)

16. A 1-kg mass, when attached to a large spring, stretches the spring a distance of 4.9 m.

(a) Calculate the spring constant.

(b) The system is placed in a viscous medium that supplies a damping constant $\mu = 3\,\text{kg/s}$. The system is allowed to come to rest. Then the mass is displaced 1 m in the downward direction and given a sharp tap, imparting an instantaneous velocity of 1 m/s in the downward direction. Find the position of the mass as a function of time and plot the solution.

17. Prove that an overdamped solution of $my'' + \mu y' + ky = 0$ can cross the time axis no more than once, regardless of the initial conditions. Use a numerical solver to create a plot of an overdamped system that crosses the time axis one time and a second plot where the plot does not cross the time axis.

18. A 50-g mass ($1\,\text{kg} = 1000\,\text{g}$) stretches a spring 20 cm ($1\,\text{m} = 100\,\text{cm}$). Find a damping constant μ so that the system is critically damped. If the mass is displaced 15 cm from its equilibrium position and released from rest, find the position of the mass as a function of time and plot the solution.

19. Prove that a critically damped solution of $my'' + \mu y' + ky = 0$ can cross the time axis no more than once, regardless of the initial conditions. Use a numerical solver to create a plot of an overdamped system that crosses the time axis one time and a second plot where the plot does not cross the time axis.

20. A spring-mass system is modeled by the equation

$$x'' + \mu x' + 4x = 0.$$

(a) Show that the system is critically damped when $\mu = 4\,\text{kg/s}$.

(b) Suppose that the mass is displaced upward 2 m and given an initial velocity of 1 m/s. Use a numerical solver to compute the solution for $\mu = 4, 4.2, 4.4, 4.6, 4.8, 5$. Plot all of the solution curves on one figure. What is special about the critically damped solution in comparison to the other solutions? Why would you want to adjust the spring on a screen door so that it was critically damped?

21. If $\mu > 2\sqrt{km}$, the system $mx'' + \mu x' + kx = 0$ is overdamped. The system is allowed to come to equilibrium. Then the mass is given a sharp tap, imparting an instantaneous downward velocity v_0.

(a) Show that the position of the mass is given by

$$x(t) = \frac{v_0}{\gamma} e^{-\mu t/(2m)} \sinh \gamma t,$$

where

$$\gamma = \frac{\sqrt{\mu^2 - 4mk}}{2m}.$$

(b) Show that the mass reaches its lowest point at

$$t = \frac{1}{\gamma} \tanh^{-1} \frac{2m\gamma}{\mu},$$

a time independent of the initial conditions.

(c) Show that, in the critically damped case, the time it takes the mass to reach its lowest point is given by $t = 2m/\mu$.

22. A 100-g mass (1 kg = 1000 g) is hung from a spring having spring constant 9.8 kg/s^2. The system is placed in a viscous medium that imparts a force of 0.3 N when the mass is moving at 0.2 m/s. Assume that the force applied by the medium is proportional, but opposite, to the mass's velocity. The mass is displaced 10 cm from its equilibrium position and released from rest. Find the amplitude, frequency, and phase of the resulting motion. Plot the solution.

23. A capacitor (0.008 F) is charged to 50 V and then connected in series with an inductor (4 H) and a resistor (20 Ω). Initially, there is no current in the circuit. Find the amplitude, frequency, and phase of the current and plot its graph.

24. A 10-kg mass stretches a spring 1 m. The system is placed in a viscous medium that provides a damping constant $\mu = 20$ kg/s. The system is allowed to attain equilibrium. Then a sharp tap to the mass imparts an instantaneous downward velocity of 1.2 m/s. Find the amplitude, frequency, and phase of the resulting motion. Plot the solution.

25. A capacitor (0.02 F) is charged to 1 V and then connected in series with an inductor (10 H) and a resistor (40 Ω). Initially, there is no current in the circuit. Find the amplitude, frequency, and phase of the charge on the capacitor and plot its graph.

4.5 Inhomogeneous Equations; the Method of Undetermined Coefficients

We now turn to the solution of inhomogeneous linear equations. These are equations of the form

$$y'' + py' + qy = f, \tag{5.1}$$

where $p = p(t)$, $q = q(t)$, and $f = f(t)$ are functions of the independent variable. Remember that f is called the inhomogeneous term, or the forcing term.

Our solution strategy comes from the understanding of the structure of the general solution, which is contained in the next theorem. We refer you to a similar result for first-order linear equations in Theorem 4.41 in Section 2.4 of Chapter 2.

THEOREM 5.2 Suppose that y_p is a particular solution to the inhomogeneous equation (5.1), and that y_1 and y_2 form a fundamental set of solutions to the associated homogeneous equation

$$y'' + py' + qy = 0. \tag{5.3}$$

Then the general solution to the inhomogeneous equation (5.1) is given by

$$y = y_p + C_1 y_1 + C_2 y_2, \tag{5.4}$$

where C_1 and C_2 are arbitrary constants. ■

Notice that the general solution can be written as $y = y_p + y_h$, where $y_h = C_1 y_1 + C_2 y_2$ is the general solution to the corresponding homogeneous equation. Thus, to find the general solution to the inhomogeneous equation (5.1) we first find

Table 1 The method of undetermined coefficients

Forcing function $f(t)$	Trial solution $y_p(t)$	Comments
e^{rt}	ae^{rt}	
$\cos \omega t$ or $\sin \omega t$	$a \cos \omega t + b \sin \omega t$	
$P(t)$	$p(t)$	P is a polynomial; p is a polynomial of the same degree
$P(t) \cos \omega t$ or $P(t) \sin \omega t$	$p(t) \cos \omega t + q(t) \sin \omega t$	P is a polynomial; p & q are polynomials of the same degree
$e^{rt} \cos \omega t$ or $e^{rt} \sin \omega t$	$e^{rt}[a \cos \omega t + b \sin \omega t]$	
$e^{rt} P(t) \cos \omega t$ or $e^{rt} P(t) \sin \omega t$	$e^{rt}[p(t) \cos \omega t + q(t) \sin \omega t]$	P is a polynomial; p & q are polynomials of the same degree

EXERCISES

In Exercises 1–4, use the technique demonstrated in Example 5.6 to find a particular solution for the given differential equation.

1. $y'' + 3y' + 2y = 4e^{-3t}$ **2.** $y'' + 6y' + 8y = -3e^{-t}$

3. $y'' + 2y' + 5y = 12e^{-t}$ **4.** $y'' + 3y' - 18y = 18e^{2t}$

In Exercises 5–8, use the form $y_p = a \cos \omega t + b \sin \omega t$, as in Example 5.8, to help find a particular solution for the given differential equation.

5. $y'' + 4y = \cos 3t$ **6.** $y'' + 9y = \sin 2t$

7. $y'' + 7y' + 6y = 3 \sin 2t$

8. $y'' + 7y' + 10y = -4 \sin 3t$

9. Suppose that $z(t) = x(t) + iy(t)$ is a solution of

$$z'' + pz' + qz = Ae^{i\omega t}. \tag{5.31}$$

Substitute $z(t)$ into equation (5.31). Then compare the real and imaginary parts of each side of the resulting equation to prove two facts:

$$x'' + px' + qx = A \cos \omega t,$$
$$y'' + py' + qy = A \sin \omega t.$$

Write a short paragraph summarizing the significance of this result.

In Exercises 10–13, use the complex method, as in Example 5.12, to find a particular solution for the differential equation.

10. $y'' + 4y = \cos 3t$ **11.** $y'' + 9y = \sin 2t$

12. $y'' + 7y' + 6y = 3 \sin 2t$

13. $y'' + 7y' + 10y = -4 \sin 3t$

In Exercises 14–17, use the technique shown in Example 5.17 to find a particular solution for the given differential equation.

14. $y'' + 5y' + 4y = 2 + 3t$ **15.** $y'' + 6y' + 8y = 2t - 3$

16. $y'' + 5y' + 6y = 4 - t^2$ **17.** $y'' + 3y' + 4y = t^3$

In Exercises 18–23, use the technique of Section 4.3 to find a solution of the associated homogeneous equation; then use the technique of this section to find a particular solution. Use Theorem 5.2 to form the general solution. Then find the solution satisfying the given initial conditions.

18. $y'' + 3y' + 2y = 3e^{-4t}, \quad y(0) = 1, \quad y'(0) = 0$

19. $y'' - 4y' - 5y = 4e^{-2t}, \quad y(0) = 0, \quad y'(0) = -1$

20. $y'' + 2y' + 2y = 2 \cos 2t, \quad y(0) = -2, \quad y'(0) = 0$

21. $y'' - 2y' + 5y = 3 \cos t, \quad y(0) = 0, \quad y'(0) = -2$

22. $y'' + 4y' + 4y = 4 - t, \quad y(0) = -1, \quad y'(0) = 0$

23. $y'' - 2y' + y = t^3, \quad y(0) = 1, \quad y'(0) = 0$

In Exercises 24–29, the forcing term is also a solution of the associated homogeneous solution. Use the technique of Example 5.20 to find a particular solution.

24. $y'' - 3y' - 10y = 3e^{-2t}$ **25.** $y'' - y' - 2y = 2e^{-t}$

26. $y'' + 4y = 4 \cos 2t$ **27.** $y'' + 9y = \sin 3t$

28. $y'' + 4y' + 4y = 2e^{-2t}$ **29.** $y'' + 6y' + 9y = 5e^{-3t}$

30. If $y_f(t)$ is a solution of

$$y'' + py' + qy = f(t)$$

and $y_g(t)$ is a solution of

$$y'' + py' + qy = g(t),$$

show that $z(t) = \alpha y_f(t) + \beta y_g(t)$ is a solution of

$$y'' + py' + qy = \alpha f(t) + \beta g(t),$$

where α and β are any real numbers.

Use the technique suggested by Examples 5.23 and 5.26, as well as Exercise 30, to help find particular solutions for the differential equations in Exercises 31–38.

31. $y'' + 2y' + 2y = 2 + \cos 2t$

32. $y'' - y = t - e^{-t}$

33. $y'' + 25y = 2 + 3t + \cos 5t$

34. $y'' + 2y' + y = 3 - e^{-t}$

35. $y'' + 4y' + 3y = \cos 2t + 3 \sin 2t$

36. $y'' + 2y' + 2y = 3 \cos t - \sin t$

37. $y'' + 4y' + 4y = e^{-2t} + \sin 2t$

38. $y'' + 16y = e^{-4t} + 3 \sin 4t$

39. Use the form $y_p(t) = (at + b)e^{-4t}$ in an attempt to find a particular solution of the equation $y'' + 3y' + 2y = te^{-4t}$.

Use an approach similar to that in Exercise 39 to find particular solutions of the equations in Exercises 40–43.

40. $y'' - 3y' + 2y = te^{-3t}$ **41.** $y'' + 2y' + y = t^2e^{-2t}$

42. $y'' + 5y' + 4y = te^{-t}$ **43.** $y'' + 3y' + 2y = t^2e^{-2t}$

44. Use the form $y_p = e^{-2t}(a \cos t + b \sin t)$ in an attempt to find a particular solution of $y'' + 2y' + 2y = e^{-2t} \sin t$.

45. If $z(t) = x(t) + iy(t)$ is a solution of

$$z'' + pz' + qz = Ae^{(a+bi)t},$$

show that $x(t)$ and $y(t)$ are solutions of

$$x'' + px' + qx = Ae^{at} \cos bt$$

and

$$y'' + py' + qy = Ae^{at} \sin bt,$$

respectively.

46. Use the technique suggested by Exercise 45 to find a particular solution of the equation in Exercise 44.

47. Prove that the imaginary part of the solution of $z'' + z' + z = te^{it}$ is a solution of $y'' + y' + y = t \sin t$. Use this idea to find a particular solution of $y'' + y' + y = t \sin t$.

48. Prove Theorem 5.22.

4.6 Variation of Parameters

In this section we introduce a technique called ***variation of parameters***. This technique is used to find a particular solution to more general higher-order equations, provided we know a fundamental set of solutions to the associated homogeneous equation. As we did in the previous section, we will illustrate the method for second-order equations. The method also works for higher-order equations, but it is usually more efficient to solve the associated first-order system using variation of parameters. This will be discussed in a later chapter.

We are interested in solving the equation

$$y'' + p(t)y' + q(t)y = g(t). \tag{6.1}$$

Notice that we are allowing the coefficients $p(t)$ and $q(t)$ to be functions of t. In particular, we are not restricting them to be constants. This might seem to be a great increase in generality, but there is a rather strong constraint. We will have to assume that we have computed a fundamental set of solutions y_1 and y_2 to the associated homogeneous equation

$$y'' + p(t)y' + q(t)y = 0. \tag{6.2}$$

Then the general solution to the homogeneous equation is

$$y_h = C_1y_1 + C_2y_2, \tag{6.3}$$

where C_1 and C_2 are arbitrary constants.

The idea behind variation of parameters is to replace the constants C_1 and C_2 in (6.3) by unknown functions $v_1(t)$ and $v_2(t)$ and look for a particular solution to the inhomogeneous equation (6.1) of the form

$$y_p = v_1y_1 + v_2y_2. \tag{6.4}$$

You will notice the similarity with the method of variation of parameters as it was used to solve first-order linear equations in Chapter 2.

(a) Differentiate y_p:

$$y_p' = (v_1'y_1 + v_2'y_2) + v_1y_1' + v_2y_2'$$

and set the first term on the right equal to zero.

$$v_1'y_1 + v_2'y_2 = 0 \tag{6.17}$$

(b) Take the second derivative of y_p and insert y_p, y_p', and y_p'' into the differential equation. After simplifying, a second equation will appear:

$$v_1'y_1' + v_2'y_2' = g(t). \tag{6.18}$$

(c) Solve (6.17) and (6.18) for v_1' and v_2' by elimination.

(d) Integrate to find v_1 and v_2.

4. Substitute v_1 and v_2 into $y_p = v_1y_1 + v_2y_2$.

It is up to the reader to decide which of the methods to use in step 3.

Although variation of parameters always works theoretically, its success requires a fundamental set of solutions to the homogeneous equation and the ability to compute the integrals in (6.16). The method is of significant theoretical importance, however.

EXERCISES

For Exercises 1–12, find a particular solution to the given second-order differential equation.

1. $y'' + 9y = \tan 3t$
2. $y'' + 4y = \sec 2t$
3. $y'' - y = t + 3$
4. $x'' - 2x' - 3x = 4e^{3t}$
5. $y'' - 2y' + y = e^t$
6. $x'' - 4x' + 4x = e^{2t}$
7. $x'' + x = \tan^2 t$
8. $x'' + x = \sec^2 t$
9. $x'' + x = \sin^2 t$
10. $y'' + 2y' + y = t^5e^{-t}$
11. $y'' + y = \tan t + \sin t + 1$
12. $y'' + y = \sec t + \cos t - 1$
13. Verify that $y_1(t) = t$ and $y_2(t) = t^{-3}$ are solutions to the homogeneous equation

$$t^2y''(t) + 3ty'(t) - 3y(t) = 0.$$

Use variation of parameters to find the general solution to

$$t^2y''(t) + 3ty'(t) - 3y(t) = \frac{1}{t}.$$

14. Verify that $y_1(t) = t^{-1}$ and $y_2(t) = t^{-1}\ln t$ are solutions to the homogeneous equation

$$t^2y''(t) + 3ty'(t) + y(t) = 0.$$

Use variation of parameters to find the general solution to

$$t^2y''(t) + 3ty'(t) + y(t) = \frac{1}{t}.$$

4.7 Forced Harmonic Motion

In this section, we apply the technique of undetermined coefficients to analyze harmonic motion with an external sinusoidal forcing term. We derived the model in Section 4.4, and with a sinusoidal forcing term, equation (4.4) becomes

$$x'' + 2cx' + \omega_0^2x = A\cos\omega t. \tag{7.1}$$

The constant A is the amplitude of the driving force, and ω is the ***driving frequency***. Remember that c is the damping constant and ω_0 is the natural frequency.

To focus our thinking, we may suppose that we have an iron mass m suspended on a spring, with the top of the spring attached to a motor that moves the top of the spring up and down. We can also consider an *RLC* circuit in which the source voltage is sinusoidal.

We will first treat the case with no damping.