EXERCISES: APPENDIX II

In Exercises 1–12, the z-plane region D consists of the complex numbers $z = x + yi$ that satisfy the given conditions. Describe (or sketch) the image R of D in the w-plane under the given function $w = f(z)$.

1. $0 \le x \le 1,\ 0 \le y \le 2; \quad w = \bar{z}.$

2. $x + y = 1; \quad w = \bar{z}.$

3. $1 \le |z| \le 2,\ \dfrac{\pi}{2} \le \arg z \le \dfrac{3\pi}{4}; \quad w = z^2.$

4. $0 \le |z| \le 2,\ 0 \le \arg(z) \le \dfrac{\pi}{2}; \quad w = z^3.$

5. $0 < |z| \le 2,\ 0 \le \arg(z) \le \dfrac{\pi}{2}; \quad w = \dfrac{1}{z}.$

6. $\dfrac{\pi}{4} \le \arg(z) \le \dfrac{\pi}{3}; \quad w = -iz.$

7. $\arg(z) = -\dfrac{\pi}{3}; \quad w = \sqrt{z}.$

8. $x = 1; \quad w = z^2.$

9. $y = 1; \quad w = z^2.$

10. $x = 1; \quad w = \dfrac{1}{z}.$

11. $-\infty < x < \infty,\ \dfrac{\pi}{4} \le y \le \dfrac{\pi}{2}; \quad w = e^z.$

12. $0 < x < \dfrac{\pi}{2},\ 0 < y < \infty; \quad w = e^{iz}.$

In Exercises 13–16, verify that the real and imaginary parts of each function $f(z)$ satisfy the Cauchy–Riemann equations, and thus find $f'(z)$.

13. $f(z) = z^2.$

14. $f(z) = z^3.$

15. $f(z) = \dfrac{1}{z}.$

16. $f(z) = e^{z^2}.$

17. Use the fact that $e^{yi} = \cos y + i \sin y$ (for real y) to show that

$$\cos y = \frac{e^{yi} + e^{-yi}}{2} \quad \text{and} \quad \sin y = \frac{e^{yi} - e^{-yi}}{2i}.$$

Exercise 16 suggests that we define complex functions

$$\cos z = \frac{e^{zi} + e^{-zi}}{2} \quad \text{and} \quad \sin z = \frac{e^{zi} - e^{-zi}}{2i},$$

as well as extend the definitions of the hyperbolic functions to

$$\cosh z = \frac{e^{z} + e^{-z}}{2} \quad \text{and} \quad \sinh z = \frac{e^{z} - e^{-z}}{2}.$$

Exercises 18–26 develop properties of these functions and relationships between them.

18. Show that $\cos z$ and $\sin z$ are periodic with period 2π, and that $\cosh z$ and $\sinh z$ are periodic with period $2\pi i$.

19. Show that $(d/dz)\sin z = \cos z$ and $(d/dz)\cos z = -\sin z$. What are the derivatives of $\sinh z$ and $\cosh z$?

20. Verify the identities $\cos z = \cosh(iz)$ and $\sin z = -i\sinh(iz)$. What are the corresponding identities for $\cosh z$ and $\sinh(z)$ in terms of cos and sin?

21. Find all complex zeros of $\cos z$ (i.e., all solutions of $\cos z = 0$).

22. Find all complex zeros of $\sin z$.

23. Find all complex zeros of $\cosh z$ and $\sinh z$.

24. Show that $\text{Re}(\cosh z) = \cosh x \cos y$ and $\text{Im}(\cosh z) = \sinh x \sin y$.

25. Find the real and imaginary parts of $\sinh z$.

26. Find the real and imaginary parts of $\cos z$ and $\sin z$.

Find the zeros of the polynomials in Exercises 27–32.

27. $P(z) = z^2 + 2iz$

28. $P(z) = z^2 - 2z + i$

29. $P(z) = z^2 + 2z + 5$

30. $P(z) = z^2 - 2iz - 1$

31. $P(z) = z^3 - 3iz^2 - 2z$

32. $P(z) = z^4 - 2z^2 + 4$

33. The polynomial $P(z) = z^4 + 1$ has two pairs of complex conjugate zeros. Find them, and hence express $P(z)$ as a product of two quadratic factors with real coefficients.

In Exercises 34–36, check that the given number z_1 is a zero of the given polynomial, and find all the zeros of the polynomial.

34. $P(z) = z^4 - 4z^3 + 12z^2 - 16z + 16; \quad z_1 = 1 - \sqrt{3}\,i.$

35. $P(z) = z^5 + 3z^4 + 4z^3 + 4z^2 + 3z + 1; \quad z_1 = i.$

36. $P(z) = z^5 - 2z^4 - 8z^3 + 8z^2 + 31z - 30;$ $z_1 = -2 + i.$

37. Show that the image of the circle $|z| = 2$ under the mapping $w = z^4 + z^3 - 2iz - 3$ winds around the origin in the w-plane four times.

where a and b are arbitrary constants. Substituting the initial conditions, we get

$$4 = x(0) = a, \quad \text{and} \quad 2 = x'(0) = 2b.$$

Thus, $a = 4$ and $b = 1$ and our solution is

$$x(t) = 4\cos 2t + \sin 2t.$$

EXERCISES

For each of the second-order differential equations in Exercises 1– 8, decide whether the equation is linear or nonlinear. If the equation in linear, state whether the equation is homogeneous or inhomogeneous.

1. $y'' + 3y' + 5y = 3\cos 2t$
2. $t^2y'' = 4y' - \sin t$
3. $t^2y'' + (1 - y)y' = \cos 2t$
4. $ty'' + (\sin t)y' = 4y - \cos 5t$
5. $t^2y'' + 4yy' = 0$
6. $y'' + 4y' + 7y = 3e^{-t}\sin t$
7. $y'' + 3y' + 4\sin y = 0$
8. $(1 - t^2)y'' = 3y$
9. In an experiment, a 2-kg mass is suspended from a spring. The displacement of the spring-mass equilibrium from the spring equilibrium is measured to be 50 cm. If the mass is then displaced 12 cm downward from its spring-mass equilibrium and released from rest, set up (but do not solve) the initial value problem that models this experiment. Assume no damping is present.
10. In an experiment, a 5-kg mass is suspended from a spring. The displacement of the spring-mass equilibrium from the spring equilibrium is measured to be 75 cm. The mass is then displaced 36 cm upward from its spring-mass equilibrium and then given a sharp downward tap, imparting an instantaneous downward velocity of 0.45 m/s. Set up (but do not solve) the initial value problem that models this experiment. Assume no damping is present.
11. Suppose that the spring mass system of Exercise 9 is suspended in a viscous solution that dampens the motion according to $R(v) = -0.05v$. Furthermore, a machine is attached to the top of the spring that shakes the spring up and down harmonically with period $T = 2$ s and amplitude $A = 0.5$ m. Assume that the spring is initially displaced 0.5 m upward from the spring-mass equilibrium by this driving force. Adjust the model in Exercise 9 to satisfy these additional constraints.
12. Suppose that the spring mass system of Exercise 10 is suspended in a viscous solution that dampens the motion according to $R(v) = -0.125v$. Furthermore, a machine is attached to the top of the spring that shakes the spring up and down harmonically with period $T = 4$ s and amplitude $A = 0.25$ m. Assume that the spring is initially displaced 0.25 m downward from the spring-mass equilibrium by this driving force. Adjust the model in Exercise 10 to satisfy these additional constraints.

In Exercises 13–16, show, by direct substitution, that the given functions $y_1(t)$ and $y_2(t)$ are solutions of the given differential equation. Then verify, again by direct substitution, that any linear combination $C_1y_1(t) + C_2y_2(t)$ of the two given solutions is also a solution.

13. $y'' - y' - 6y = 0, \quad y_1(t) = e^{3t}, \quad y_2(t) = e^{-2t}$
14. $y'' + 4y = 0, \quad y_1(t) = \cos 2t, \quad y_2(t) = \sin 2t$
15. $y'' - 2y' + 2y = 0, \quad y_1(t) = e^t\cos t, \quad y_2(t) = e^t\sin t$
16. $y'' + 4y' + 4y = 0, \quad y_1(t) = e^{-2t}, \quad y_2(t) = te^{-2t}$

In Exercises 17–20, use Definition 1.22 to explain why $y_1(t)$ and $y_2(t)$ are linearly independent solutions of the given differential equation. In addition, calculate the Wronskian and use it to explain the independence of the given solutions.

17. $y'' - y' - 2y = 0, \quad y_1(t) = e^{-t}, \quad y_2(t) = e^{2t}$
18. $y'' + 9y = 0, \quad y_1(t) = \cos 3t, \quad y_2(t) = \sin 3t$
19. $y'' + 4y' + 13y = 0, \quad y_1(t) = e^{-2t}\cos 3t, \quad y_2(t) = e^{-2t}\sin 3t$
20. $y'' + 6y' + 9y = 0, \quad y_1(t) = e^{-3t}, \quad y_2(t) = te^{-3t}$
21. Show that the functions

$$y_1(t) = t^2 \quad \text{and} \quad y_2(t) = t|t|$$

are linearly independent on $(-\infty, +\infty)$. Next, show that the Wronskian of the two functions is identically zero on the interval $(-\infty, +\infty)$. Why doesn't this result contradict Proposition 1.27?

22. Show that $y_1(t) = e^t$ and $y_2(t) = e^{-3t}$ form a fundamental set of solutions for $y'' + 2y' - 3y = 0$, then find a solution satisfying $y(0) = 1$ and $y'(0) = -2$.
23. Show that $y_1(t) = \cos 4t$ and $y_2(t) = \sin 4t$ form a fundamental set of solutions for $y'' + 16y = 0$, then find a solution satisfying $y(0) = 2$ and $y'(0) = -1$.
24. Show that $y_1(t) = e^{-t}\cos 2t$ and $y_2(t) = e^{-t}\sin 2t$ form a fundamental set of solutions for $y'' + 2y' + 5y = 0$, then find a solution satisfying $y(0) = -1$ and $y'(0) = 0$.
25. Show that $y_1(t) = e^{-4t}$ and $y_2(t) = te^{-4t}$ form a fundamental set of solutions for $y'' + 8y' + 16y = 0$, then find a solution satisfying $y(0) = 2$ and $y'(0) = -1$.

The solution is discussed in Propositions 3.3, 3.11, and 3.18.

This information can be summarized as follows:

- If $p^2 - 4q > 0$, the characteristic equation has two distinct, real roots λ_1 and λ_2. A fundamental set of solutions is
$$y_1(t) = e^{\lambda_1 t} \quad \text{and} \quad y_2(t) = e^{\lambda_2 t}.$$
- If $p^2 - 4q = 0$, the characteristic equation has one repeated real root λ. A fundamental set of solutions is
$$y_1(t) = e^{\lambda t} \quad \text{and} \quad y_2(t) = te^{\lambda t}.$$
- If $p^2 - 4q < 0$, the characteristic equation has two complex conjugate roots $a \pm ib$. A fundamental set of solutions is
$$y_1(t) = e^{at}\cos bt \quad \text{and} \quad y_2(t) = e^{at}\sin bt.$$

EXERCISES

The equations in Exercises 1–8 have distinct, real, characteristic roots. Find the general solution in each case.

1. $y'' - y' - 2y = 0$
2. $2y'' - 3y' - 2y = 0$
3. $y'' + 5y' + 6y = 0$
4. $y'' + y' - 12y = 0$
5. $2y'' - y' - y = 0$
6. $6y'' + y' - y = 0$
7. $3y'' - 2y' - y = 0$
8. $6y'' + 5y' - 6y = 0$

The equations in Exercises 9–16 have complex characteristic roots. Find the general solution in each case.

9. $y'' + y = 0$
10. $y'' + 4y = 0$
11. $y'' + 4y' + 5y = 0$
12. $y'' + 2y' + 17y = 0$
13. $y'' + 2y = 0$
14. $y'' + 2y' + 3y = 0$
15. $y'' - 2y' + 4y = 0$
16. $y'' + 2y' + 2y = 0$

The equations in Exercises 17–24 have repeated, real, characteristic roots. Find the general solution in each case.

17. $y'' - 4y' + 4y = 0$
18. $y'' - 6y' + 9y = 0$
19. $4y'' + 4y' + y = 0$
20. $4y'' + 12y' + 9y = 0$
21. $16y'' + 8y' + y = 0$
22. $y'' + 4y' + 4y = 0$
23. $16y'' + 24y' + 9y = 0$
24. $y'' + 8y' + 16y = 0$

In Exercises 25–36, find the solution of the given initial value problem.

25. $y'' - y' - 2y = 0, \quad y(0) = -1, \quad y'(0) = 2$
26. $10y'' - y' - 3y = 0, \quad y(0) = 1, \quad y'(0) = 0$
27. $y'' - 2y' + 17y = 0, \quad y(0) = -2, \quad y'(0) = 3$
28. $y'' + 25y = 0, \quad y(0) = 1, \quad y'(0) = -1$
29. $y'' + 10y' + 25y = 0, \quad y(0) = 2, \quad y'(0) = -1$
30. $y'' - 2y' - 3y = 0, \quad y(0) = 2, \quad y'(0) = -3$
31. $y'' + 2y' + 3y = 0, \quad y(0) = 1, \quad y'(0) = 0$
32. $y'' - 4y' - 5y = 0, \quad y(1) = -1, \quad y'(1) = -1$
33. $8y'' + 2y' - y = 0, \quad y(-1) = 1, \quad y'(-1) = 0$
34. $4y'' + y = 0, \quad y(1) = 0, \quad y'(1) = -2$
35. $y'' + 12y' + 36y = 0, \quad y(1) = 0, \quad y'(1) = -1$
36. $y'' - 4y' + 13y = 0, \quad y(0) = 4, \quad y'(0) = 0$

37. Suppose that λ_1 and λ_2 are characteristic roots of the characteristic equation $\lambda^2 + p\lambda + q = 0$, where p and q are real constants.
(a) Prove that $\lambda_1\lambda_2 = q$.
(b) Prove that $\lambda_1 + \lambda_2 = -p$.

38. Given that the characteristic equation $\lambda^2 + p\lambda + q = 0$ has a double root, $\lambda = \lambda_1$, show, by direct substitution, that $y = te^{\lambda_1 t}$ is a solution of $y'' + py' + qy = 0$.

4.4 Harmonic Motion

In Section 4.1 we derived the equation for the motion of a vibrating spring. It is (see equation (1.13))

$$my'' + \mu y' + ky = F(t), \tag{4.1}$$

where the constant coefficients are the mass m, the damping constant μ, and the spring constant k, and the function $F(t)$ is an external force.