

# TMA4115 Matematikk 3: Revision Lecture

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# Lay Chapters 1-4

## ① Chapter One:

Solving systems of linear equations by row reduction, conversion to matrices and vectors, solution sets, linear independence, linear transformations.

## ② Chapter Two:

Working with matrices, inverses.

## ③ Chapter Three:

Determinants.

## ④ Chapter Four:

Vector spaces, subspaces, null space, column space, bases, dimension, rank, Markov chains.

# Linear Systems

$$\begin{array}{rcl} 3x_2 + 2x_3 & = & 1 \\ 2x_1 + 4x_3 & = & -2 \\ x_1 + 2x_2 + 3x_3 & = & 0 \end{array}$$

# Linear Systems

$$\begin{array}{rcl} & 3 & + 2 = 1 \\ 2 & & + 4 = -2 \\ & + 2 & + 3 = 0 \end{array}$$

# Linear Systems

$$\begin{array}{ccc} & 3 & 2 & 1 \\ 2 & & 4 & -2 \\ & 2 & 3 & 0 \end{array}$$

# Linear Systems

$$\begin{array}{cccc} 0 & 3 & 2 & 1 \\ 2 & 0 & 4 & -2 \\ 1 & 2 & 3 & 0 \end{array}$$

# Linear Systems

$$\begin{bmatrix} 0 & 3 & 2 & 1 \\ 2 & 0 & 4 & -2 \\ 1 & 2 & 3 & 0 \end{bmatrix}$$

# Linear Systems

$$\left[ \begin{array}{ccc|c} 0 & 3 & 2 & 1 \\ 2 & 0 & 4 & -2 \\ 1 & 2 & 3 & 0 \end{array} \right]$$

Solve  $\begin{aligned} 3x_2 + 2x_3 &= 1 \\ 2x_1 + 4x_3 &= -2 \\ x_1 + 2x_2 + 3x_3 &= 0 \end{aligned}$   $\iff$  Row reduce  $\left[ \begin{array}{cccc} 0 & 3 & 2 & 1 \\ 2 & 0 & 4 & -2 \\ 1 & 2 & 3 & 0 \end{array} \right]$

# Solve *Everything* with Gauss

Question Asks For	Gaussian Answer
Solution of linear system	Row reduce augmented matrix, read off solution
Number of solutions	RRAM, free columns?
Matrix injective (one-to-one)	Row reduce coefficient matrix, free columns?
Any solutions	RRAM, $0 \dots 0 \ 1$ ?
Matrix surjective (onto)	RRCM, zero rows?
Basis for column space	RRCM, pivot columns <b>from original matrix</b>
Basis for row space	RRCM, non-zero rows
Basis for null space	RRCM, set each free variable to <b>1</b>
Determinant	RRCM, keep track of scales and swaps
Invertible	RRCM, identity?

# Solve *Everything* with Gauss

$$[A|b] = \left[ \begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 1 & 3 & 5 & 1 \\ 2 & 2 & 2 & -2 \end{array} \right]$$

Basis  $\text{Col}(A)$  linearly independent

Rank = 2

Basis  $\text{Row}(A)$  not onto

$\det(A) = 0$

not 1-1 linearly dependent

$\dim \text{Null } A = 1$

Solution exists



$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 0 & -3 & 4 \\ 0 & 2 & 5 & -2 \end{bmatrix}$$

→

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -5 & 2 \\ 0 & 2 & 5 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ a-1 \end{bmatrix}$$

$$SL(A) = \left\{ \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\} \right\}$$

$$x_3 = 0 \quad x_4 = 1$$

$$Null(A) = \left\langle \left\{ \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \\ 0 \\ 1 \end{bmatrix} \right\} \right\rangle$$

$\begin{aligned} -2x_2 - 5x_3 + 2x_4 &= 0 \\ \Rightarrow x_2 &= 1 \\ x_1 + x_2 + x_3 + x_4 &= 0 \\ x_1 + 1 + 0 + 1 &= 0 \end{aligned}$

$\left\{ \begin{array}{l} Ax = b \\ \end{array} \right.$

$$\left[ \begin{array}{cccc|c} 1 & 1 & -1 & 1 & 1 \\ 2 & 0 & -3 & 4 & 1 \\ 0 & 2 & 5 & -2 & a \end{array} \right]$$

$$\left[ \begin{array}{c} 1 \\ 1 \\ a \end{array} \right] \rightarrow \left[ \begin{array}{c} 1 \\ -1 \\ a \end{array} \right] \rightarrow \left[ \begin{array}{c} 1 \\ -1 \\ a-1 \end{array} \right] \Rightarrow \text{Solution}$$

$\xleftarrow[a=1]{}$

Intuition:  $Ax = b$  always has a solution

In this case,  $Ax = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$  has no solution  
 $\Rightarrow$  not onto

I-1:  $Ax = b$  always has at most one solution

$\Leftrightarrow Ax = 0$  has exactly one solution

In this case, free variables  $\Rightarrow$  not I-1

$$\begin{bmatrix} 1 & 2 & a \\ 3 & 7 & b \\ 2 & 9 & c \end{bmatrix}$$

Things we know about invertible matrices

1/  $\det(A) \neq 0 \Leftrightarrow$  invertible

2/ must be square

3/ linear independence of columns + square  $\Leftrightarrow$

4/ row equivalent to identity  $\Leftrightarrow$

5/ sq + onto  $\Leftrightarrow$

6/ Sq + 1-1  $\Leftrightarrow$

7/ Sq + (Ax=b always has sol)  $\Leftrightarrow$

8/ Sq + (Ax=b has unique sol)  $\Leftrightarrow$

9)  $\text{sq} + \text{ref}$  has pivots in every col/row  $\Leftrightarrow$

$$\begin{bmatrix} 1 & 2 & a \\ 3 & 7 & b \\ 2 & 9 & c \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & a \\ 0 & 1 & b - 3a \\ 0 & 5 & c - 2a \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & a \\ 0 & 1 & b - 3a \\ 0 & 0 & c - 2a - 5(b - 3a) \end{bmatrix}$$

$$= 13a - 5b + c$$

Invertible  $\Leftrightarrow 13a - 5b + c \neq 0.$

$$\det(A) = 13a - 5b + c$$

Finding an inverse (without determinants)

$$\left[ \begin{array}{ccc|ccc} 1 & 2 & a & 1 & 0 & 0 \\ 3 & 7 & b & 0 & 1 & 0 \\ 2 & 9 & c & 0 & 0 & 1 \end{array} \right]$$

$$\rightarrow \left[ \begin{array}{cccccc|ccc} 1 & 2 & a & 1 & 0 & 0 & 1 & 0 & 7a-2b & 7-2 \\ 0 & 1 & b-3a & -3 & 1 & 0 & 0 & 1 & b-3a & -3 \\ 0 & 5 & c-2a & -2 & 0 & 1 & 0 & 0 & 13a-5b+c & 13-5 \end{array} \right] \rightarrow \left[ \begin{array}{cccccc|ccc} 1 & 0 & 7a-2b & 7-2 & 0 & 0 & 1 & 0 & b-3a & -3 \\ 0 & 1 & b-3a & -3 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 13a-5b+c & 13-5 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right]$$

to get integers: need  $(13a-5b+c)^{-1} \in \mathbb{Z}$  since  $(3, 5)-term$

$$\rightarrow (13a-5b+c)^{-1}$$

Assume  $a, b, c \in \mathbb{Z}$ , then  $13a-5b+c \in \mathbb{Z}$ , so  $13a-5b+c$  must be  $\pm 1$ . Then for any  $a, b$ , set  $c = 5b - 13a \pm 1$  to get a solution. Eg  $a=b=0, c=1$ , or  $a=1, b=-1, c=-19$ .

$$\det \begin{vmatrix} 0 & 1 \\ -1 & 2 \end{vmatrix} = 1$$