Use Gaussian Elimination

2010 v6	bases & orth proj	20	
2009 h 4, 5	— " —, lin sys + rank	5	15
2009 k 4, 6	lin sys + param, theoretical	$\boxed{27}$	$\boxed{35}$
2009 v5	lin sys + bases orth	5	
2008 h 4	theory + lin ind	20	

2009 k6 a) A square, A inv $\Leftrightarrow A^TA$ inv

b) A $m \times n$, $Null(A) = Null(A^TA)$

a) A inv $\Rightarrow A^TA$ inv & A^TA inv $\Rightarrow A$ inv

A inv — $\det A \neq 0$

$\boxed{\exists B : BA = I,\ AB = I}$

$rref(A) = I$

$rank A = size A$

$Ax = 0$ unique sol^n

$Null(A) = \{0\}$

$Ax = b$ has sol^n all b

$(BA = I,\ AB = I)^T = (A^TB^T = I^T,\ B^TA^T = I^T)$

$\Rightarrow A^T$ is invertible

Product of inv matrices is inv

$\Rightarrow A^TA$ inv

A^TA inv — $[\exists C : C(A^TA) = I,\ (A^TA)C = I]$

A inv $\Rightarrow$ $\det(A) \neq 0$

$\det(A^T) = \det(A)$ so $\det(A^T) \neq 0$

$\det(A^TA) = \det(A^T)\det(A) \neq 0$

so A^TA inv

(slightly easier to reverse)

$\det(A^TA) = \det(A)^2$

b) $Ax = 0$ apply A^T: $A^TAx = A^T0 = 0$

$Ax = 0 \Rightarrow A^TAx = 0$

$\left(x^TBy = (B^Tx)^Ty \right)$

$\frac{16}{64} = \frac{1}{4}$

$A^TA\underline{x} = \underline{0}$

$\underline{x}^TA^TA = \underline{0}^T$

$\underline{\underline{\underline{x}^TA^TA\underline{x} = 0}}$

$y^Ty = \|y\|^2 = \sum y_i^2$

$y = Ax$

y^Ty so $y = \underline{0}$, ie $A\underline{x} = \underline{0}$

$A^TA\underline{x} = \underline{0} \Rightarrow A\underline{x} = \underline{0}$

2009 k4

$$A = \begin{bmatrix} a & 0 & 1 & 0 \\ 0 & a & 0 & 1 \\ 1 & 0 & b & 0 \\ 0 & 1 & 0 & b \end{bmatrix} \qquad \underline{b} = \begin{bmatrix} 1 \\ 1 \\ 4 \\ 4 \end{bmatrix}$$

a) which a, b does $A\underline{x} = \underline{b}$ have ∞ - sol$^{\underline{n}}$s?

b) how does rank A vary with a & b?

c) why is A ~~...~~ diag all a & b? What is char pol A when $a = b = 1$?

a)
$$\left[\begin{array}{cccc|c} a & 0 & 1 & 0 & 1 \\ 0 & a & 0 & 1 & 1 \\ 1 & 0 & b & 0 & 4 \\ 0 & 1 & 0 & b & 4 \end{array}\right] \quad \left[\begin{array}{cccc|c} 1 & 0 & b & 0 & 4 \\ 0 & 1 & 0 & b & 4 \\ a & 0 & 1 & 0 & 1 \\ 0 & a & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 - aR_1} \left[\begin{array}{cccc|c} 1 & 0 & b & 0 & 4 \\ 0 & 1 & 0 & b & 4 \\ 0 & 0 & 1-ab & 0 & 1-4a \\ 0 & a & 0 & 1 & 1 \end{array}\right] \xrightarrow{R_4 - aR_2} \left[\begin{array}{cccc|c} 1 & 0 & b & 0 & 4 \\ 0 & 1 & 0 & b & 4 \\ 0 & 0 & 1-ab & 0 & 1-4a \\ 0 & 0 & 0 & 1-ab & 1-4a \end{array}\right]$$

$x_1 + bx_3 = 4$

$x_2 + bx_4 = 4$

$(1-ab)x_3 = 1-4a$

$(1-ab)x_4 = 1-4a$

have x_3, x_4 free iff $ab = 1$.

then have a solution iff $1-4a = 0$

$a = 1/4$

Inf solutions iff $a = 1/4$, $b = 4$.

b) if $ab = 1$, $\text{rank}(A) = 2$

$ab \neq 1$, $\text{rank}(A) = 4$

so A is inv iff $ab \neq 1$.

2010 v 6

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & 2 & 1 \\ 3 & 6 & 1 & 0 & 2 & -1 \\ 4 & 8 & 2 & -2 & 0 & -4 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 6 \\ 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 2 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -3 & -4 & -4 \\ 0 & 0 & 2 & -6 & -8 & -8 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & \boxed{0} & 1 & 2 & 1 \\ 0 & 0 & 1 & -3 & -4 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

nice

- Col(A): cols of A corr to pivot cols in rref(A)

$$\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \ \& \ \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

- Null(A): $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 4 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

- Row(A): non-zero rows in rref(A) ~~or~~ their corr rows in A

$$\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -3 \\ -4 \\ -4 \end{bmatrix}$$

- $Col(A)^\perp = Null(A^T)$ so run GE on A^T.

Methods

- Make basis orthogonal and use $\sum \frac{v^T b_i}{b_i^T b_i} b_i$
- $A^T A x = A^T b$ then answer is Ax
- Find another matrix B with $\underline{col(B) = col(A)}$ and solve $B^T B x = B^T b$

$$\begin{bmatrix} 1 & 0 \\ 3 & 1 \\ 4 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 1 \\ 4 & 2 \end{bmatrix} x = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 26 & 11 \\ 11 & 5 \end{bmatrix} x = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \qquad \begin{bmatrix} 26 & 11 & 4 \\ 11 & 5 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} & \\ & \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \rightsquigarrow \left\{ \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} - \frac{11}{26} \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} -11 \\ -7 \\ 8 \end{bmatrix} \right\}$$

$$\frac{4}{26} \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} + \frac{8}{234} \begin{bmatrix} -11 \\ -7 \\ 8 \end{bmatrix} = \begin{bmatrix} \\ \end{bmatrix}$$