TMA4115 Matematikk 3

Andrew Stacey

Norges Teknisk-Naturvitenskapelige Universitet
Trondheim

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Lecture 25: The Best of All Possible Worlds

Andrew Stacey

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Trondheim

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Key Points

- Orthogonality + Diagonalisation = Symmetry
- Comes from Calculus
- Leads to Topology

Recap

- Basis of eigenvectors reveals structure
- Eigenvectors “natural” directions
- Partial information of eigenvalues classifies behaviour of system

Orthogonal Eigenvectors

ODE + Matrices + Orthogonality

$$\longrightarrow$$

Orthogonal Eigenvectors!

Question

When does A have a basis of orthogonal eigenvectors?

Spectral Theorem

Answer

A has a basis of orthogonal eigenvectors

$$\Longleftrightarrow$$

$$A^\top = A.$$

Definition

$$A^\top = A$$

$$\Longleftrightarrow$$

A *is*

symmetric

Quick Check in 2D: $\Longrightarrow$

$$\left\{\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} -b \\ a \end{bmatrix}\right\}, \quad a^2 + b^2 = 1$$

$$A\begin{bmatrix} a \\ b \end{bmatrix} = \lambda \begin{bmatrix} a \\ b \end{bmatrix}, A\begin{bmatrix} -b \\ a \end{bmatrix} = \mu \begin{bmatrix} -b \\ a \end{bmatrix}$$

$$A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = aA\begin{bmatrix} a \\ b \end{bmatrix} - bA\begin{bmatrix} -b \\ a \end{bmatrix} = \begin{bmatrix} \lambda a^2 + \mu b^2 \\ (\lambda - \mu)ab \end{bmatrix}$$

$$A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = bA\begin{bmatrix} a \\ b \end{bmatrix} + aA\begin{bmatrix} -b \\ a \end{bmatrix} = \begin{bmatrix} (\lambda - \mu)ab \\ \mu a^2 + \lambda b^2 \end{bmatrix}$$

Quick Check in 2D: $\Longleftarrow$

$$A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$$

$$c(\lambda) = \lambda^2 - (a+d)\lambda + (ad - b^2)$$

$$\frac{(a+d) \pm \sqrt{a^2 + 2ad + d^2 - 4ad + 4b^2}}{2}$$

Conclusion: Always at least one real eigenvalue

Quick Check in 2D: $\Longleftarrow$

Suppose

$$\begin{bmatrix} a & b \\ b & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ bx + dy \end{bmatrix}$$

$$-ayx + bx^2 - by^2 + dxy = \begin{bmatrix} x \\ y \end{bmatrix} \bullet \alpha \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix} \bullet \beta \begin{bmatrix} -y \\ x \end{bmatrix} = \alpha$$

Conclusion:

$$\begin{bmatrix} a & b \\ b & d \end{bmatrix}\begin{bmatrix} -y \\ x \end{bmatrix} = 0 \begin{bmatrix} x \\ y \end{bmatrix} + \beta \begin{bmatrix} -y \\ x \end{bmatrix}$$

Symmetric Matrices from Functions

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x, y)$$

Partial derivatives

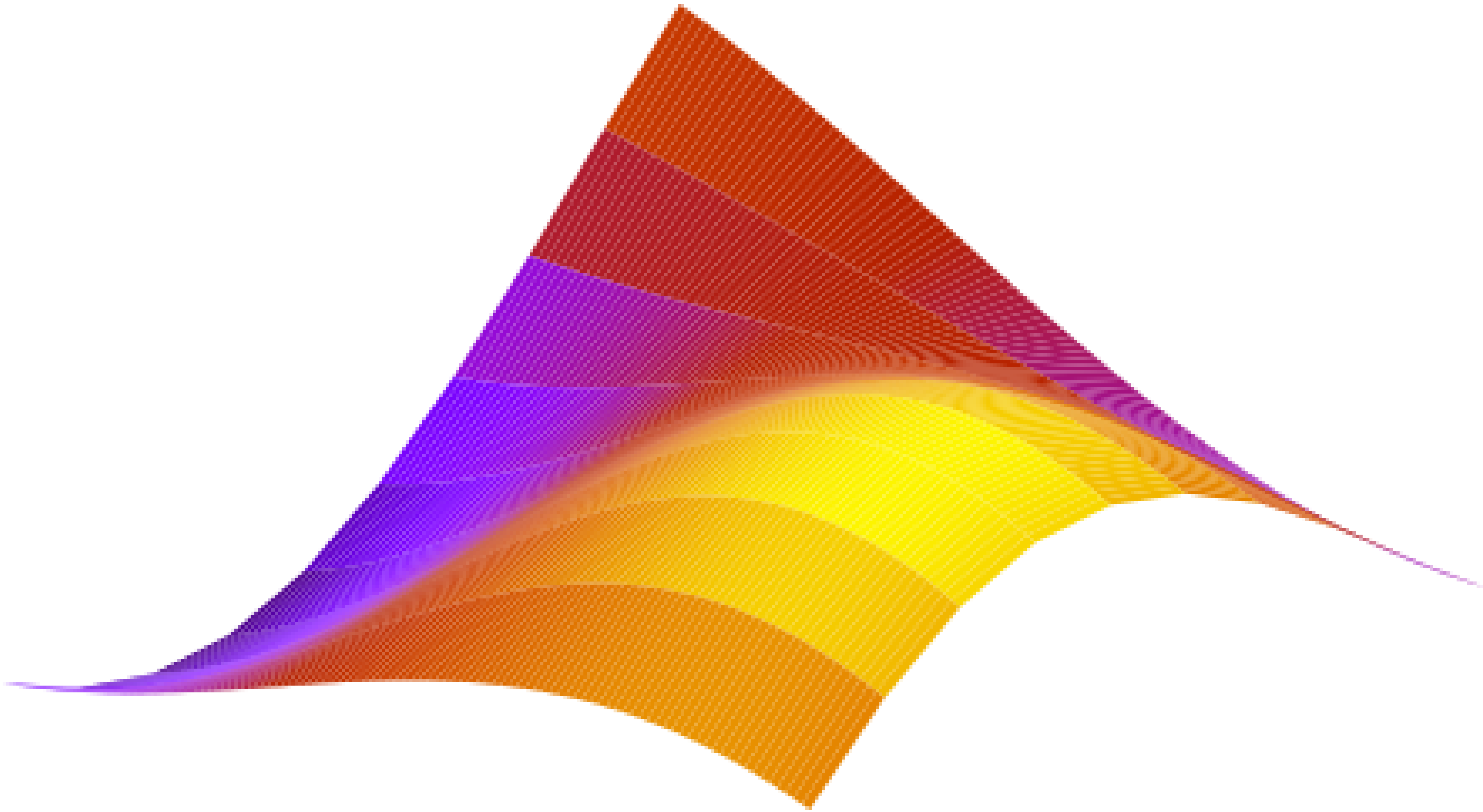
$$\frac{\partial f}{\partial x}(x, y) := \lim_{h \to 0} \frac{1}{h}\Big(f(x + h, y) - f(x, y)\Big)$$

Gradient

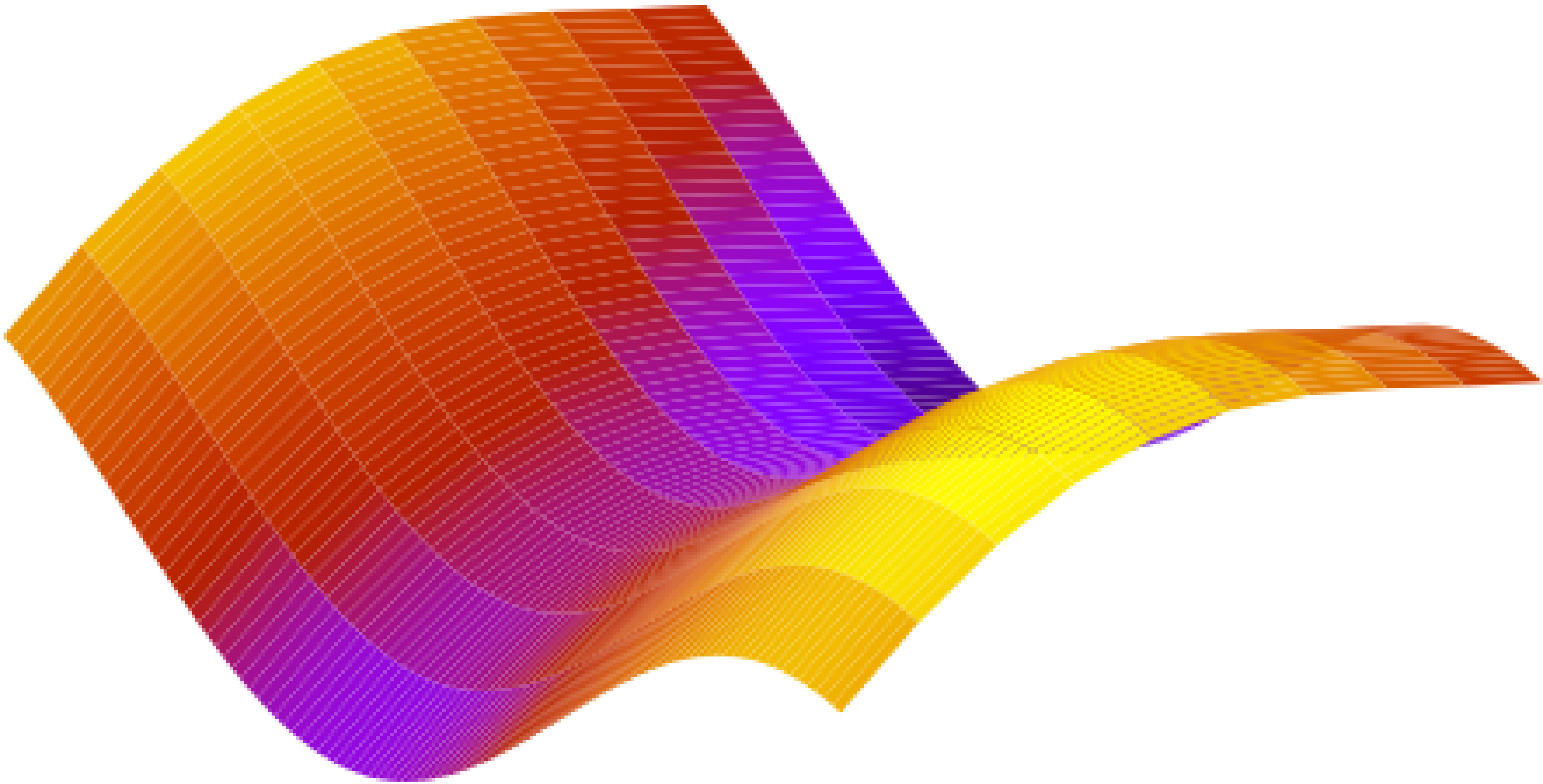
$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix}$$

Points in the direction of greatest change in f

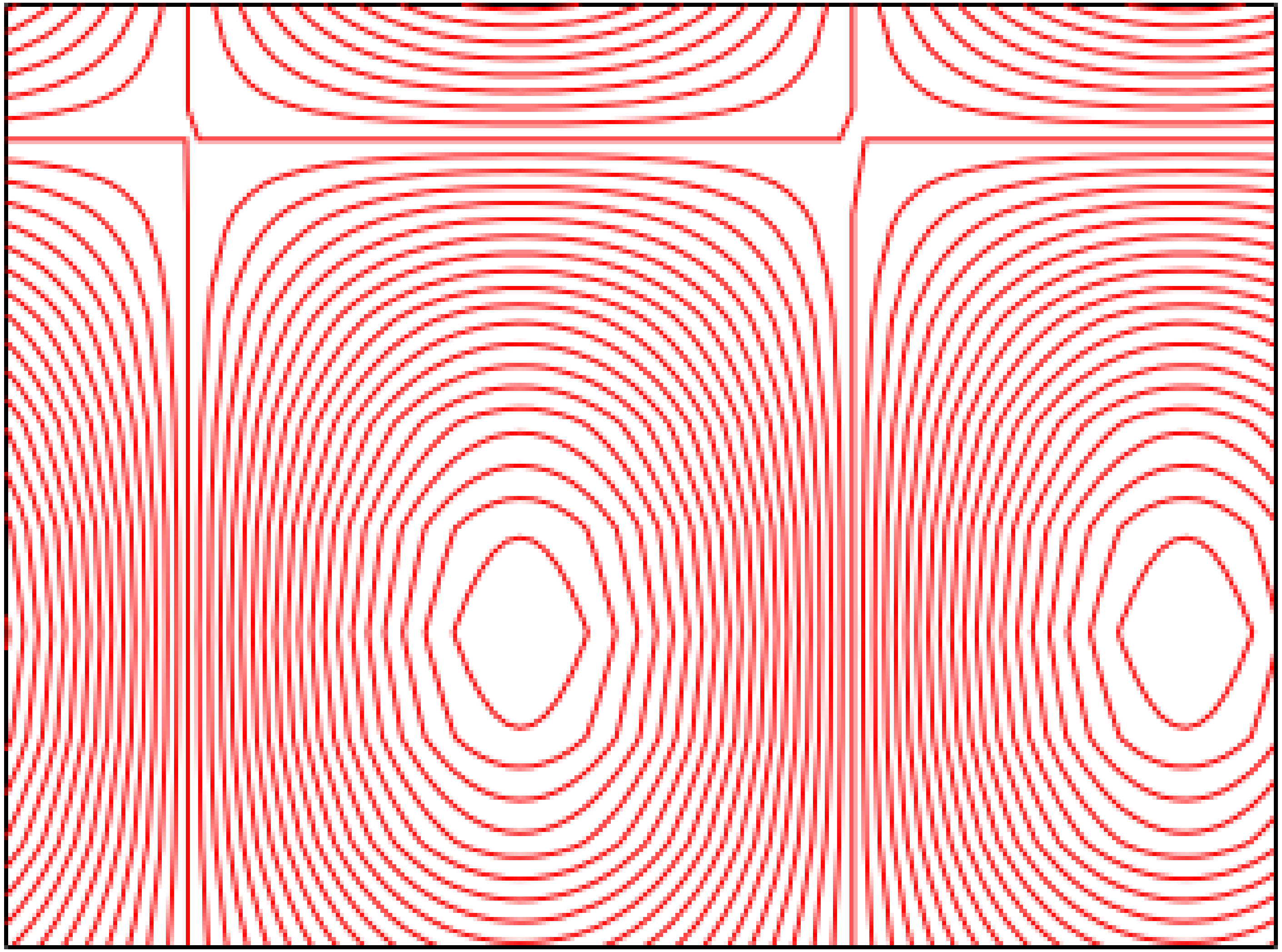
Some Examples



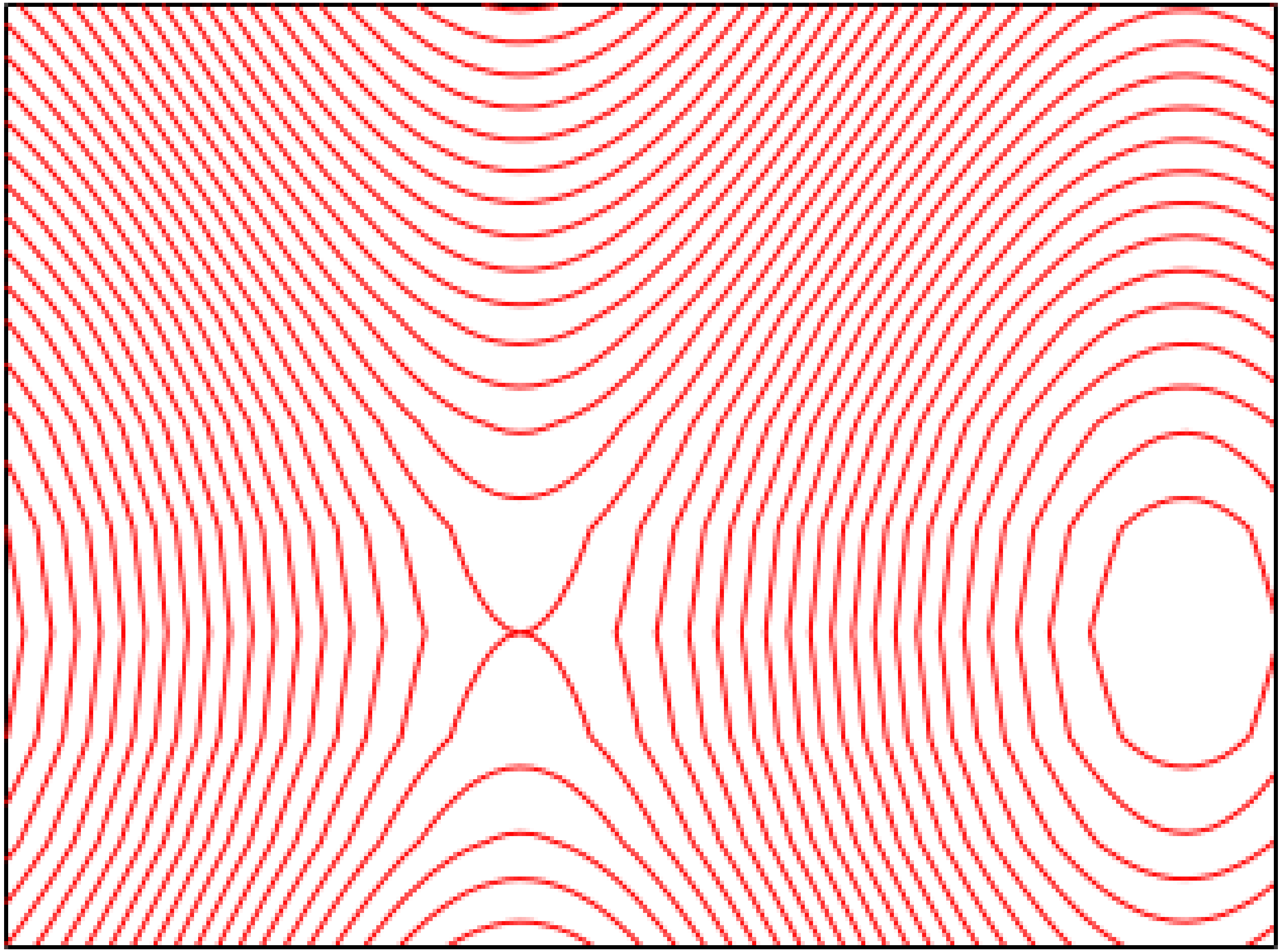
Some Examples



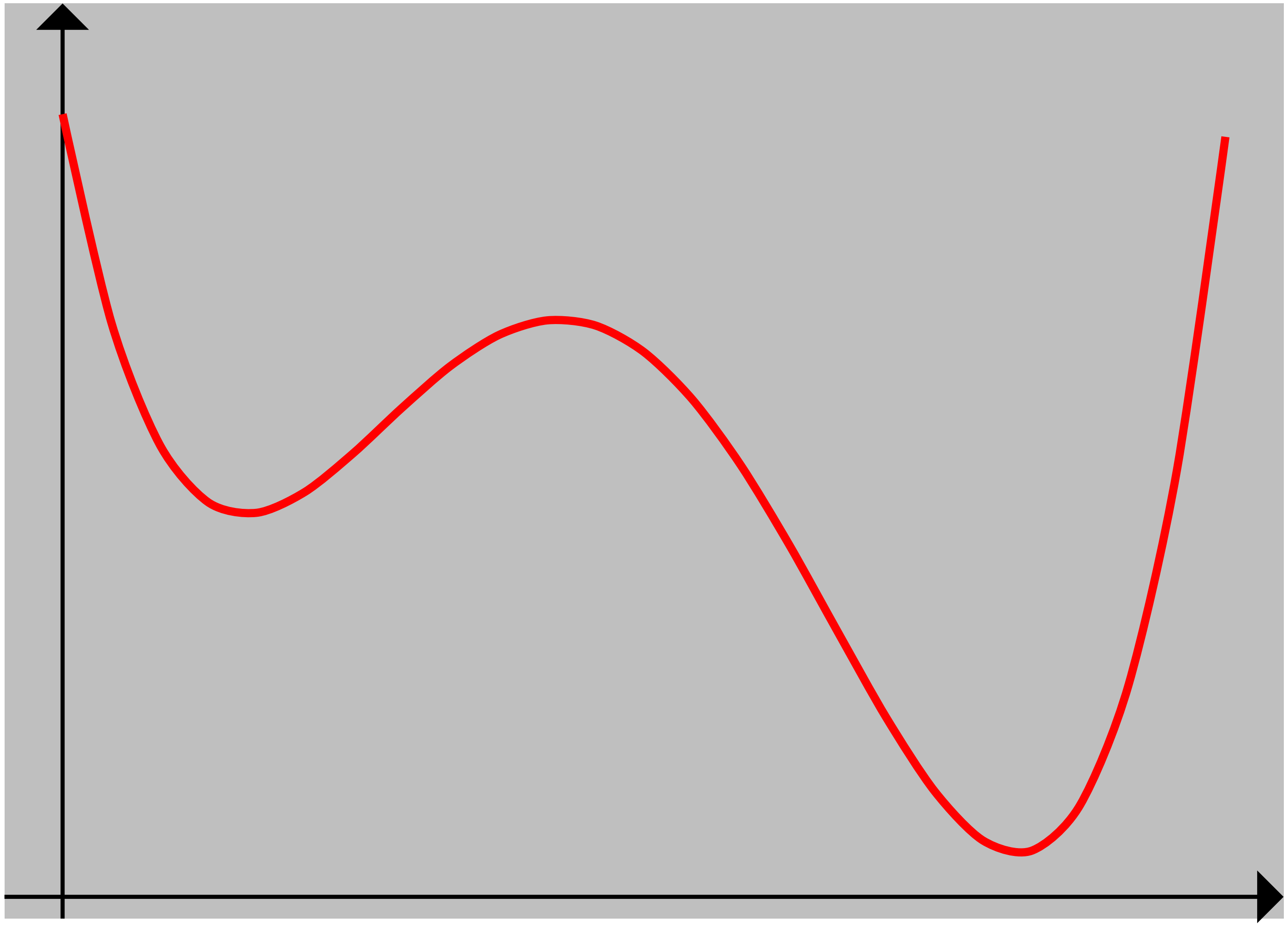
Some Examples



Some Examples



Interesting In Parts



View from Far Away

Remark

The interesting parts of a function are where it changes direction.

Reminder

$g\colon \mathbb{R} \to \mathbb{R}$

- Critical Point: $g'(t_0) = 0$
- $g''(t_0) > 0 \implies$ minimum
- $g''(t_0) < 0 \implies$ maximum
- $g''(t_0) = 0 \implies$ need more information

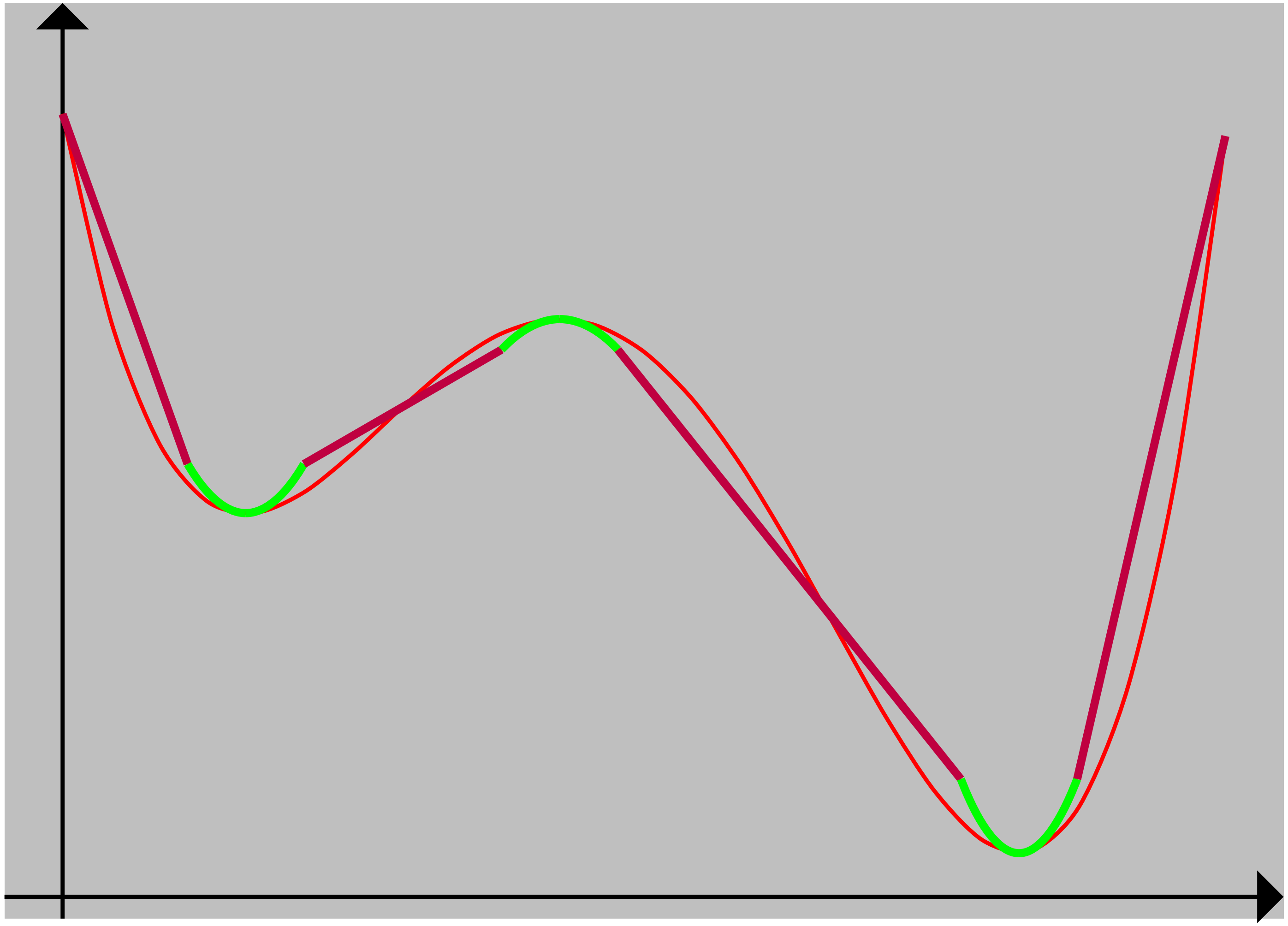
In Close-Up

$g'(t_0) = 0 \Longrightarrow$

$$g(t_0 + h) \simeq g(t_0) + h^2 g''(t)$$

If g is “good”, put a quadratic at each critical point and join up via straight lines.

Joined-Up Functions



Higher Dimensions

First Approximation

$$f(x+h, y+k) \simeq f(x,y) + h\frac{\partial f}{\partial x}(x,y) + k\frac{\partial f}{\partial y}(x,y)$$

Second Approximation

From: $$\frac{\partial f}{\partial x}(x+h, y+k) \simeq \frac{\partial f}{\partial x} + h\frac{\partial^2 f}{\partial x^2} + k\frac{\partial^2 f}{\partial y \partial x}$$

Get: $$f(x+h, y+k) \simeq f + h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} + h^2\frac{\partial^2 f}{\partial x^2} + hk\frac{\partial^2 f}{\partial y \partial x} + k^2\frac{\partial^2 f}{\partial y^2} + kh\frac{\partial^2 f}{\partial y \partial x}$$

Critical Points

If $\nabla f(x, y) = \mathbf{0}$

$$f \simeq f(x, y) + h^2 \frac{\partial^2 f}{\partial x^2} + 2hk \frac{\partial^2 f}{\partial x \partial y} + k^2 \frac{\partial^2 f}{\partial y^2}$$

Connection with matrices

$$h \frac{\partial f}{\partial x} + k \frac{\partial f}{\partial y} = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} \bullet \begin{bmatrix} h \\ k \end{bmatrix}$$

$$h^2 \frac{\partial^2 f}{\partial x^2} + 2hk \frac{\partial^2 f}{\partial x \partial y} + k^2 \frac{\partial^2 f}{\partial y^2} = \begin{bmatrix} h \\ k \end{bmatrix} \bullet \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix}$$

Quadratic Forms

Near a "good" critical point, $f\colon \mathbb{R}^2 \to \mathbb{R}$ behaves like:

$$\begin{bmatrix} x \\ y \end{bmatrix} \bullet \begin{bmatrix} a & b \\ b & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = ax^2 + 2bxy + dy^2$$

Definition

$$ax^2 + 2bxy + dy^2$$

is a quadratic form

Orthogonal Diagonalisation

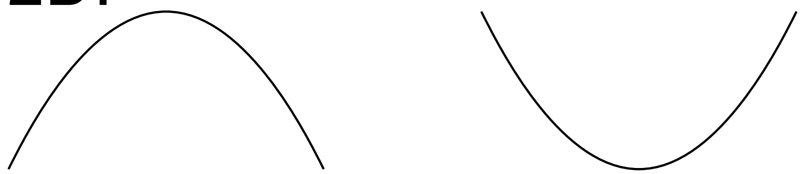
$$A = A^\top, A\mathbf{u} = \mu\mathbf{u}, A\mathbf{v} = \nu\mathbf{v}, \mathbf{u} \bullet \mathbf{v} = 0$$

$$(\alpha\mathbf{u}) \bullet A(\alpha\mathbf{u}) = (\alpha\mathbf{u}) \bullet \mu(\alpha\mathbf{u}) = \mu\alpha^2$$

$$(\alpha\mathbf{u} + \beta\mathbf{v}) \bullet A(\alpha\mathbf{u} + \beta\mathbf{v}) = \mu\alpha^2 + \nu\beta^2$$

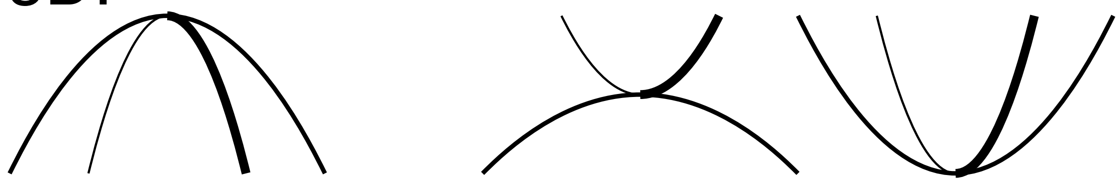
Possibilities

- 2D:



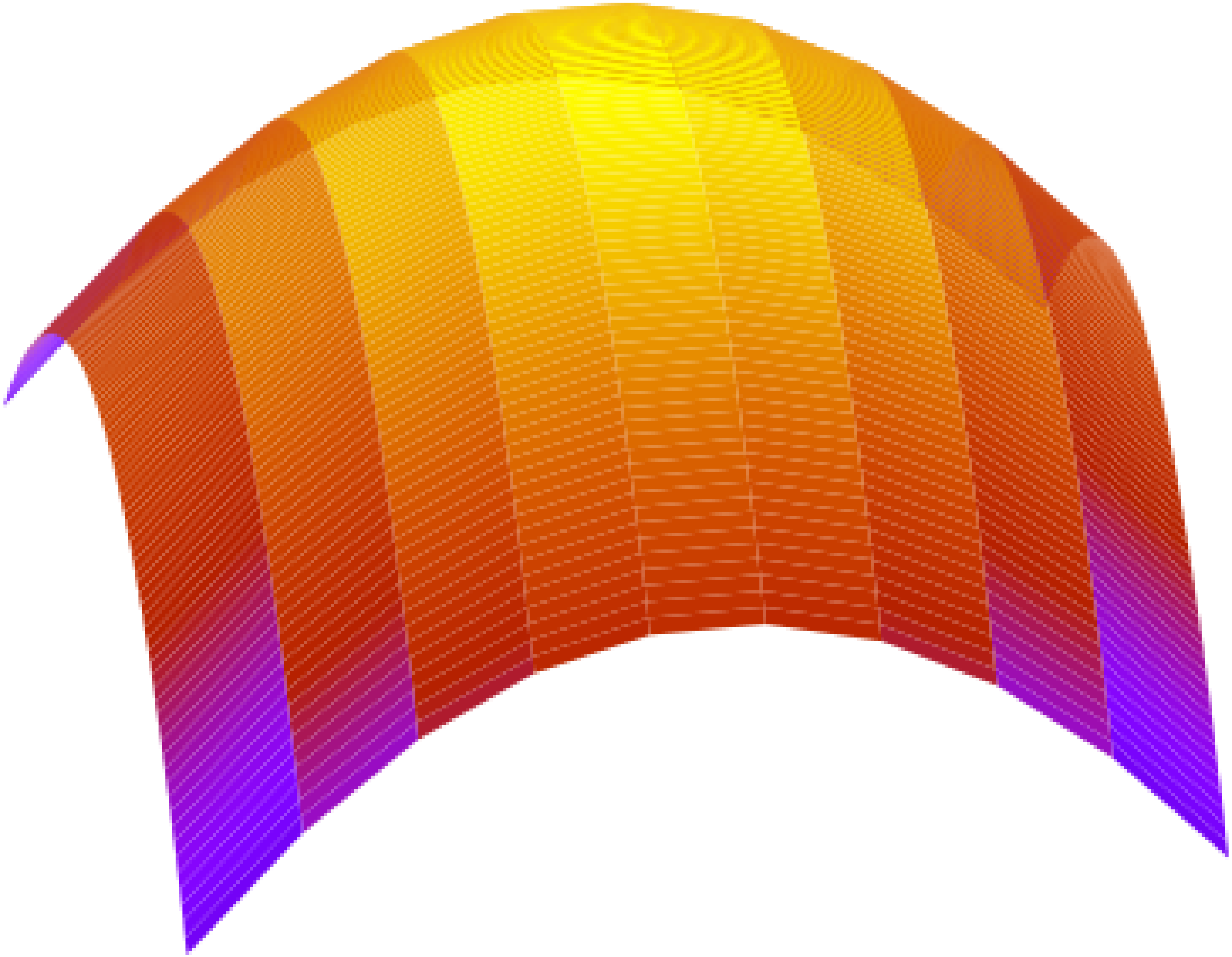
$-x^2$ $\quad$ x^2

- 3D:

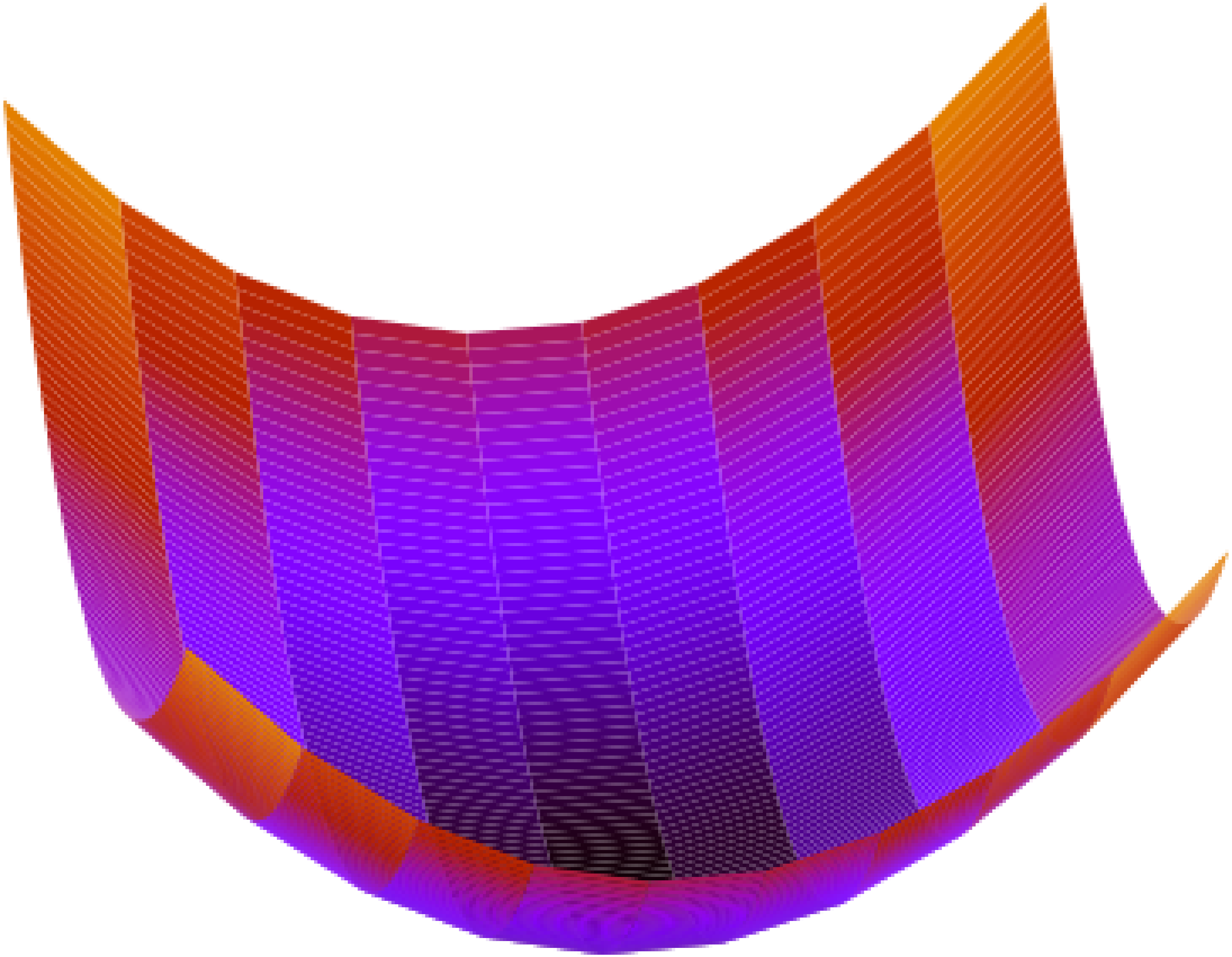


$-x^2 - y^2$ $\quad$ $-x^2 + y^2$ $\quad$ $x^2 + y^2$

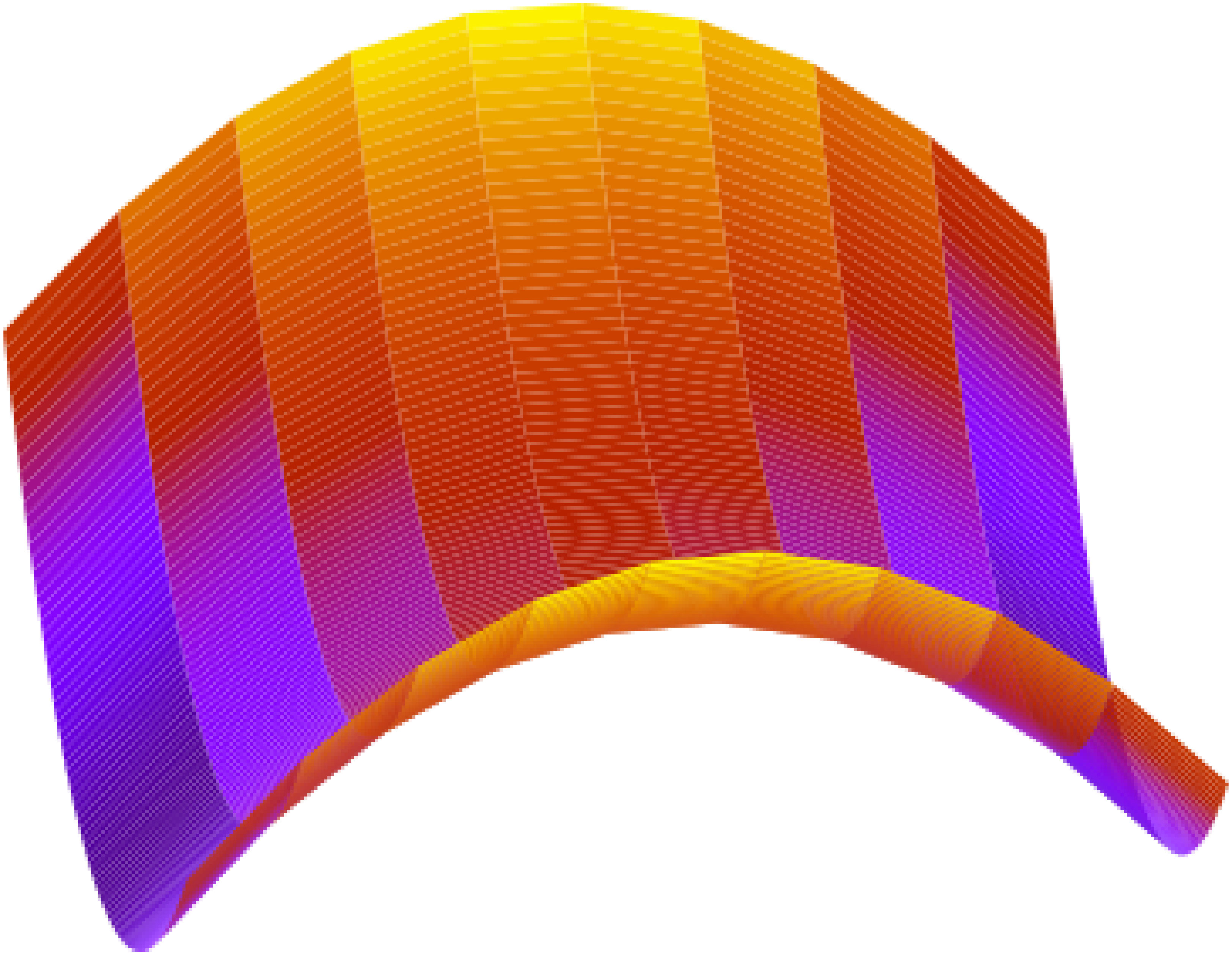
Maximum



Minimum



Saddle Point

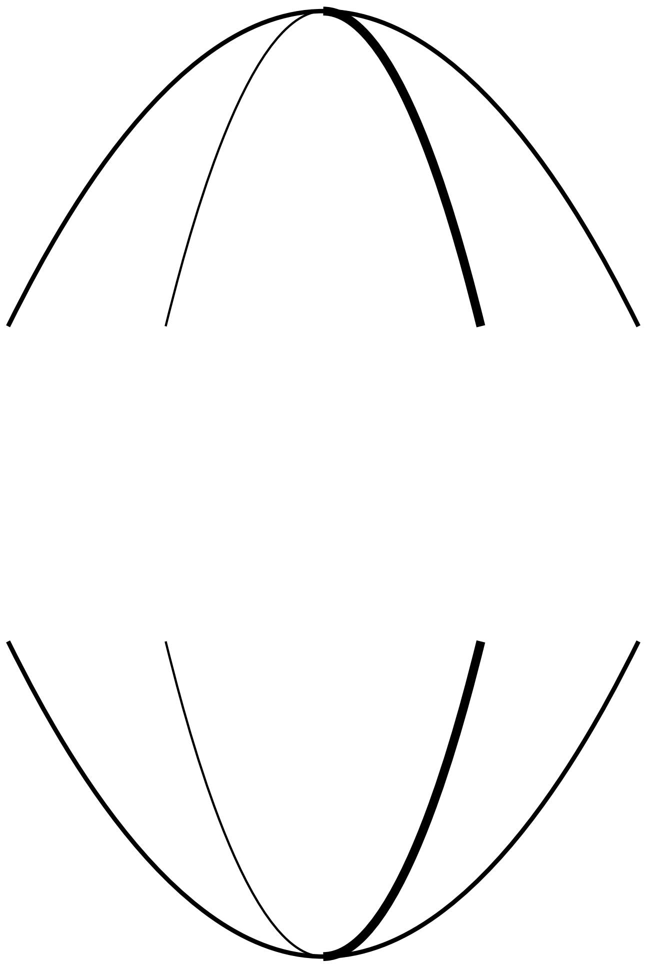


Reconstructing Surfaces

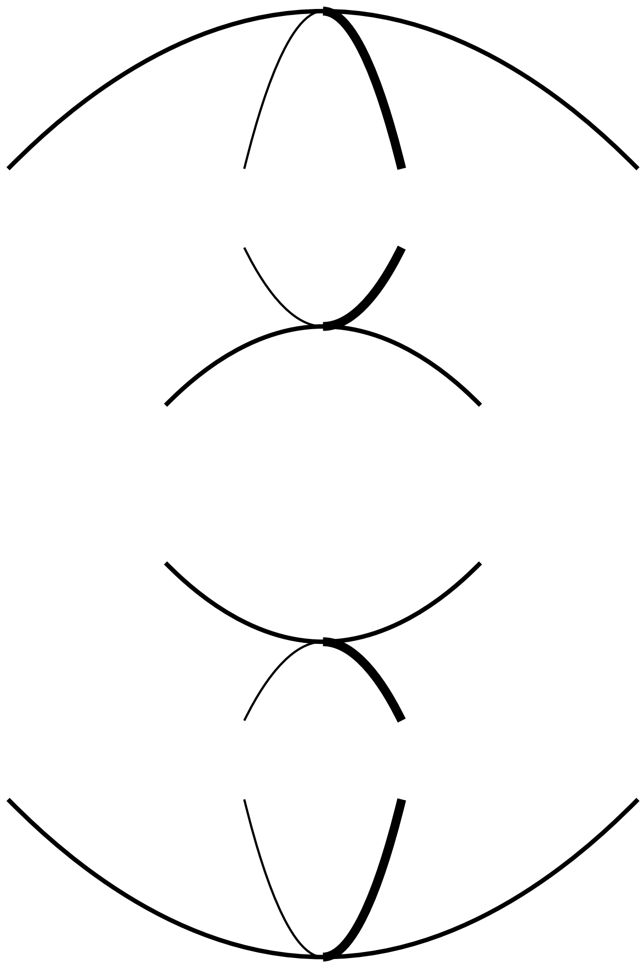
$$S \subseteq \mathbb{R}^3 \xrightarrow{f} \mathbb{R}$$

- Look at points where $\nabla f = \mathbf{0}$
- Classify by number of negative eigenvalues of $\left[\partial^2 f\right]$
- Stick together

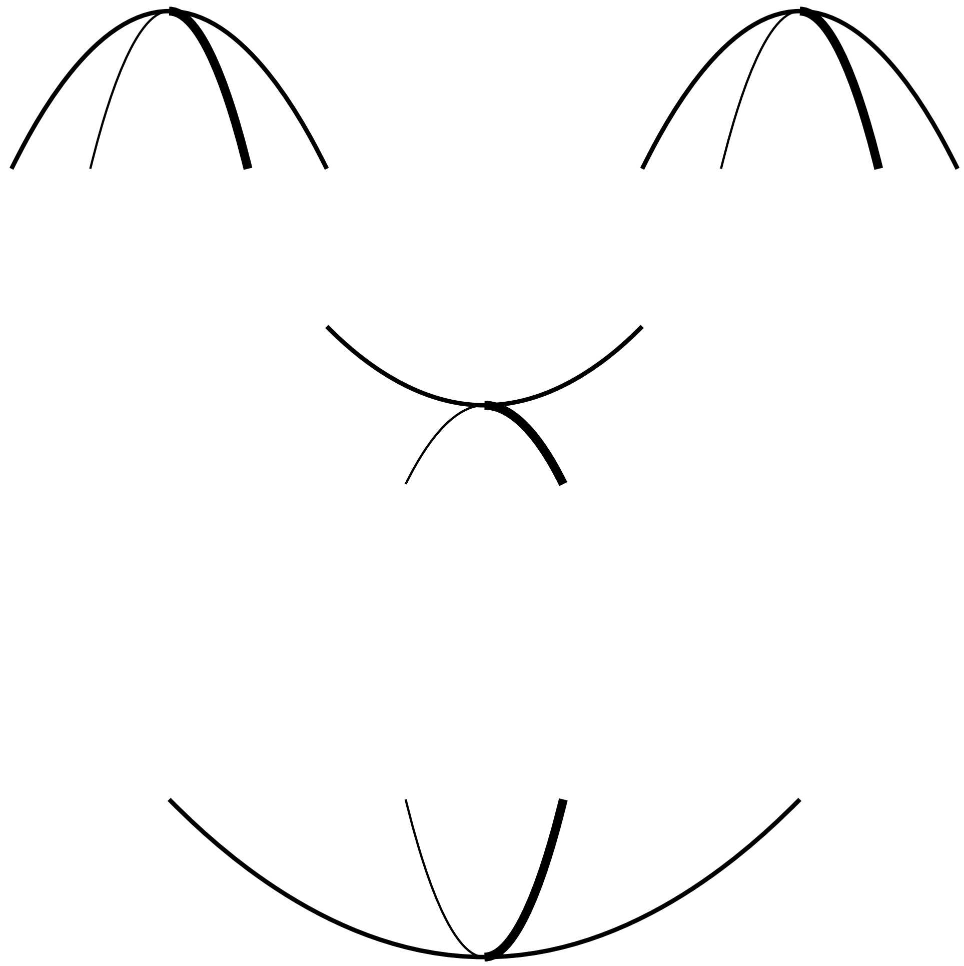
Sphere



Torus



Deformed Sphere



Which is Which

$$S \subseteq \mathbb{R}^3 \xrightarrow{f} \mathbb{R}$$

- Look at points where $\nabla f = \mathbf{0}$
- Classify by number of negative eigenvalues of $\left[\partial^2 f\right]$
- Count: $\#2 - \#1 + \#0$
  - Sphere: $1 - 0 + 1 = 2$
  - Torus: $1 - 2 + 1 = 0$
  - Deformed Sphere: $2 - 1 + 1 = 2$
  - Two-Holed Torus: $1 - 4 + 1 = -2$

Classification

This number classifies (certain) surfaces completely!

And . . .

That's what I do!

Summary

- Symmetric $\iff$ orthogonally diagonalisable
- Symmetric from differentiation
- Leads to ... Topology!