TMA4115 Matematikk 3

Andrew Stacey

Norges Teknisk-Naturvitenskapelige Universitet
Trondheim

Spring 2010

Lecture 16: Who Lives Here?

Andrew Stacey

Norges Teknisk-Naturvitenskapelige Universitet
Trondheim

5th March 2010

Key Points

- Subspaces $\longleftrightarrow$ "Rooms in the house"
- Describe subspaces using vectors
- "Linearly independent" $\Longrightarrow$ minimal description

Recap

- Processes need two **sets**:
  1. Set of valid inputs
  2. Set of possible outputs
- For a matrix, these are $\mathbb{R}^k$
- Can add, subtract, and scale vectors just like in $\mathbb{R}$
- Use vectors and structure to unlock matrices

Some Special Vectors

Introducing. . .

$$\mathbf{e}_j = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \longleftarrow j\text{th}$$

The elementary vectors.

Special Property One

Lemma

Every vector is a unique sum of scales of elementary vectors.

Proof.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + x_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$= x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n$$

□

Examples

$$\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix} + 2\begin{bmatrix}0\\1\\0\end{bmatrix} + 3\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$\begin{bmatrix}\pi\\ \mathrm{e}\\0\\2\end{bmatrix} = \pi\begin{bmatrix}1\\0\\0\\0\end{bmatrix} + \mathrm{e}\begin{bmatrix}0\\1\\0\\0\end{bmatrix} + 0\begin{bmatrix}0\\0\\1\\0\end{bmatrix} + 2\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

Special Property Two

Lemma

A matrix is completely determined by what it does to the elementary vectors.

Proof.

$$
\begin{aligned}
A\mathbf{x} &= A(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n) \\
&= x_1A\mathbf{e}_1 + x_2A\mathbf{e}_2 + \cdots + x_nA\mathbf{e}_n
\end{aligned}
$$

□

Example

$$
\begin{aligned}
\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \left(x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \\
&= x \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= x \begin{bmatrix} 1 \\ 3 \end{bmatrix} + y \begin{bmatrix} 2 \\ 4 \end{bmatrix}
\end{aligned}
$$

Unlocking Matrices

The First Key

$$A \longleftrightarrow \left\{A\mathbf{e}_1, A\mathbf{e}_2, \dots, A\mathbf{e}_n\right\}$$

The First Lock

Actual outputs versus Potential outputs

- Potential outputs: $\mathbb{R}^m$
- Actual outputs: contained within $\mathbb{R}^m$

The Question

Describe the actual outputs within the potential outputs.

Describing Actuality

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

Potential Outputs

- Anything in $\mathbb{R}^3$

Actual Outputs

$$A \begin{bmatrix} x \\ y \end{bmatrix} = x \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = x \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + y \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

Everything that can be written in the form

$$x \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + y \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

Actual Outputs

$$b = Ax \Longleftrightarrow b = x\begin{bmatrix}1\\3\\5\end{bmatrix} + y\begin{bmatrix}2\\4\\6\end{bmatrix} \text{ some } x, y \in \mathbb{R}$$

Example

1. $\begin{bmatrix}0\\0\\0\end{bmatrix} = 0\begin{bmatrix}1\\3\\5\end{bmatrix} + 0\begin{bmatrix}2\\4\\6\end{bmatrix}$

2. $\begin{bmatrix}1\\3\\5\end{bmatrix} = 1\begin{bmatrix}1\\3\\5\end{bmatrix} + 0\begin{bmatrix}2\\4\\6\end{bmatrix}$

3. $\begin{bmatrix}2\\4\\6\end{bmatrix} = 0\begin{bmatrix}1\\3\\5\end{bmatrix} + 1\begin{bmatrix}2\\4\\6\end{bmatrix}$

4. $\begin{bmatrix}0\\2\\4\end{bmatrix} = 2\begin{bmatrix}1\\3\\5\end{bmatrix} - 1\begin{bmatrix}2\\4\\6\end{bmatrix}$

Exclusive Property

$$b = Ax \Longleftrightarrow b = x\begin{bmatrix}1\\3\\5\end{bmatrix} + y\begin{bmatrix}2\\4\\6\end{bmatrix} \text{ some } x, y \in \mathbb{R}$$

Example

Can't get $\begin{bmatrix}0\\0\\1\end{bmatrix}$:

$$x\begin{bmatrix}1\\3\\5\end{bmatrix} + y\begin{bmatrix}2\\4\\6\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix} \Longrightarrow \begin{aligned} x + 2y &= 0 \\ 3x + 4y &= 0 \\ 5x + 6y &= 1 \end{aligned}$$

which can't happen

Describing Actual Outputs

$$b = Ax \iff b = x \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + y \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \text{ some } x, y \in \mathbb{R}$$

Actual outputs of $A \longleftrightarrow$ everything that can be generated from $\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$ by scaling and adding.

All linear combinations of $\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$.

Linear Combinations

Definition

a
linear combination
of
$\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^n$
is a vector $\mathbf{u}$ *for which*
there exist
$a_1, \ldots, a_k \in \mathbb{R}$
such that

$$\mathbf{u} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k$$

Remark

Note "there exist": may not be unique.

Spanning the Divide

Definition

The set of
all linear combinations
of $\mathbf{v}_1, \dots, \mathbf{v}_k$ is call their
span.

Notation: $\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$, $\langle \mathbf{v}_1, \dots, \mathbf{v}_k \rangle$

Actual output of $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ is $\text{Span}\left(\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \right)$

Actual output of A is $\text{Span}\left(A\mathbf{e}_1, A\mathbf{e}_2, \dots, A\mathbf{e}_n\right)$

Story So Far

- Potential Output: $\mathbb{R}^m$
- Actual Output: $\text{Span}\left(A\mathbf{e}_1, A\mathbf{e}_2, \ldots, A\mathbf{e}_n\right)$

Question

What's the advantage of potential output over actual output?

Possible Answers

1. Easier to use
2. More structure

Inherited Structure

Lemma

If $\mathbf{u}, \mathbf{w} \in \text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$ *and* $a \in \mathbb{R}$ *then:*

$$\mathbf{u} + \mathbf{w} \in \text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$$

and

$$a\mathbf{u} \in \text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$$

Proof.

$$\begin{bmatrix} 3 \\ 7 \\ 11 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix} = \left(\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \right) + \left(2 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} - \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \right) = (1+2) \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + (2-1) \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

$\square$

Inherited Structure

Conclusion

$\text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$ inherits addition and scaling from $\mathbb{R}^n$
In addition, they obey the same rules ...
because they hold in $\mathbb{R}^n$!

Definition

A
subspace
of $\mathbb{R}^n$ is a subset
$$W \subseteq \mathbb{R}^n$$
such that
$$\mathbf{0} \in W$$
$$\mathbf{u}, \mathbf{v} \in W \implies \mathbf{u} + \mathbf{w} \in W$$
$$\mathbf{w} \in W, a \in \mathbb{R} \implies a\mathbf{w} \in W$$

Examples

1. $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : x = y \right\}$

   1.1 $0 = 0 \Longrightarrow \mathbf{0} \in W$

   1.2 $\begin{bmatrix} 2 \\ 2 \end{bmatrix} \in W$, $3 \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix} \in W$.

   1.3 $\begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ -4 \end{bmatrix} \in W$, $\begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} -4 \\ -4 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix} \in W$.

2. $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : x^2 = y^2 \right\}$

   2.1 $0^2 = 0^2 \Longrightarrow \mathbf{0} \in W$

   2.2 $\begin{bmatrix} 2 \\ -2 \end{bmatrix} \in W$, $3 \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 6 \\ -6 \end{bmatrix} \in W$ $(6^2 = (-6)^2)$.

   2.3 $\begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ -4 \end{bmatrix} \in W$, $\begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ -4 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \end{bmatrix} \notin W$.

Examples

3. $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : x^2 + y^2 = -1 \right\}$

   3.1 $0^2 + 0^2 = 0 \neq -1 \implies \mathbf{0} \notin W$

   3.2 (in fact, $W = \varnothing$)

4. $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : (x - y)^2 = 0 \right\}$

   4.1 $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : x = y \right\}$ so subspace

One Thing, Many Views

Back to matrices...

Actual outputs: $\text{Span}\left(A\mathbf{e}_1, \ldots, A\mathbf{e}_n\right)$ subspace

Example

$$W = \text{Span}\left(\begin{bmatrix}1\\3\\5\end{bmatrix}, \begin{bmatrix}2\\4\\6\end{bmatrix}\right)$$

Question

$$\text{Matrix} \xrightarrow{\text{Apply to } \mathbf{e}_j} \text{Vectors} \xrightarrow{\text{Take span}} \text{Subspace}$$

Can we reverse those?

Describing a Subspace

$$W = \text{Span}\left(\mathbf{v}_1, \ldots, \mathbf{v}_k\right)$$

Explicit description, so very concrete. But not unique.

$$\text{Span}\left(\begin{bmatrix}1\\3\\5\end{bmatrix}, \begin{bmatrix}2\\4\\6\end{bmatrix}\right) = \text{Span}\left(\begin{bmatrix}1\\3\\5\end{bmatrix}, \begin{bmatrix}0\\2\\4\end{bmatrix}\right)$$

Which is best? Hard to tell.

Describing a Subspace

$$\text{Span}\left(\begin{bmatrix}1\\3\\5\end{bmatrix}, \begin{bmatrix}2\\4\\6\end{bmatrix}\right) = \text{Span}\left(\begin{bmatrix}1\\3\\5\end{bmatrix}, \begin{bmatrix}2\\4\\6\end{bmatrix}, \begin{bmatrix}0\\2\\4\end{bmatrix}\right)$$

Which is best? The first!

Why? Second has redundancies.

Can throw out, say, $\begin{bmatrix}0\\2\\4\end{bmatrix}$ without changing subspace.

Minimality: Linear Independence

Definition

Say that

$\mathbf{v}_1, \ldots, \mathbf{v}_k$

is

linearly independent

if throwing any out changes the span.

$$\text{Span}\left(\begin{bmatrix}1\\3\\5\end{bmatrix}, \begin{bmatrix}2\\4\\6\end{bmatrix}\right) = \text{Span}\left(\begin{bmatrix}1\\3\\5\end{bmatrix}, \begin{bmatrix}2\\4\\6\end{bmatrix}, \begin{bmatrix}0\\2\\4\end{bmatrix}\right)$$

So $\left\{\begin{bmatrix}1\\3\\5\end{bmatrix}, \begin{bmatrix}2\\4\\6\end{bmatrix}, \begin{bmatrix}0\\2\\4\end{bmatrix}\right\}$ not linearly independent.

(we say linearly dependent)

Linear Independence

Is $\left\{ \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \right\}$ linearly independent?

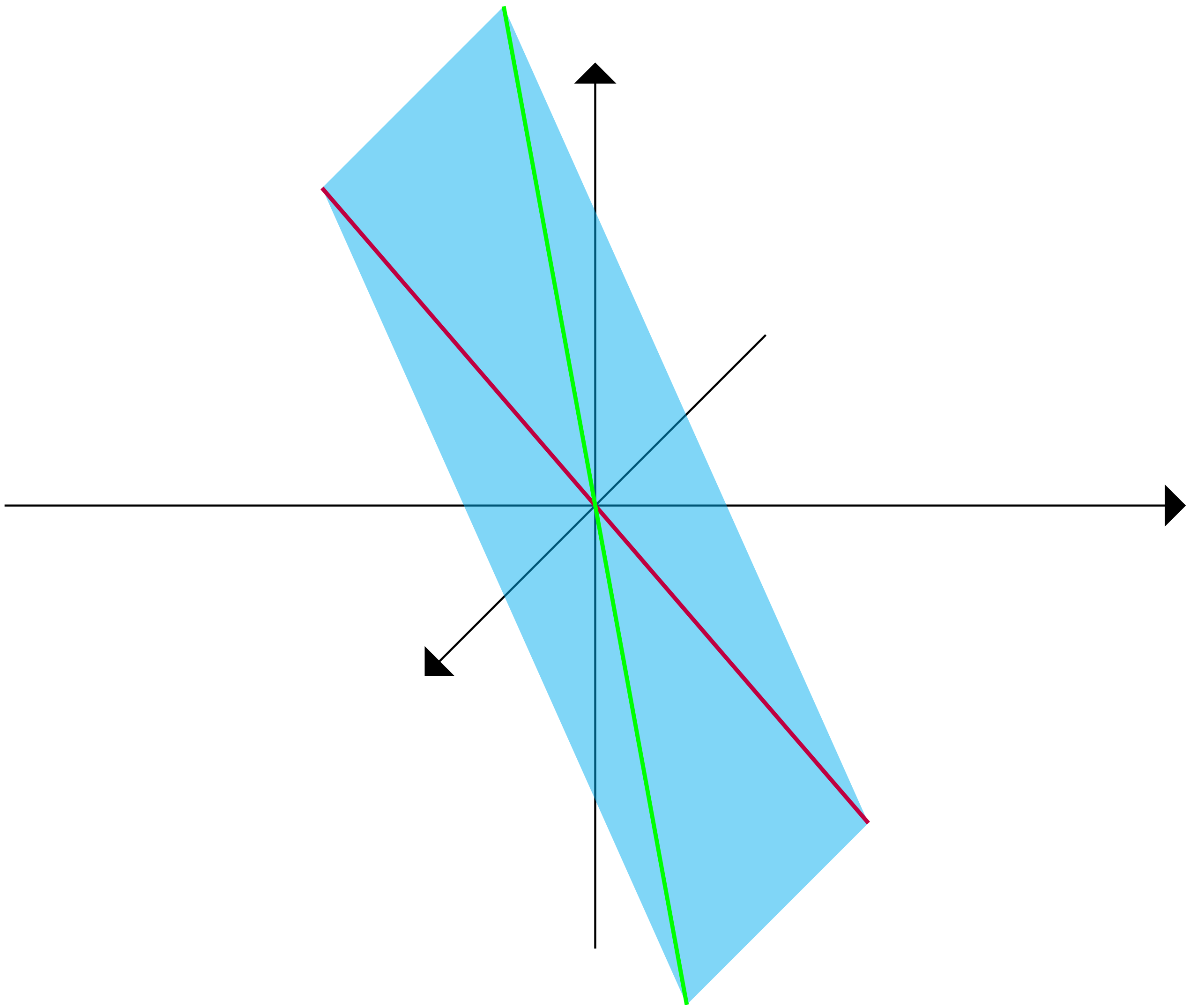
Throw some out and check span:

Is $\text{Span}\left(\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \right) = \text{Span}\left(\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \right)$? No. Done? No.

Is $\text{Span}\left(\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \right) = \text{Span}\left(\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \right)$? No. Done? Yes.

$\Longrightarrow$ Linearly independent.

In Pictures



Easier Test

Lemma

$$\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \textit{ linearly independent}$$

$$\Longleftrightarrow$$

the only solution of

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k = \mathbf{0}$$

is

$$a_1 = a_2 = \cdots = a_k = 0$$

Proof.

Suppose $\text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_k) = \text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_{k-1})$

Then there are $a_1, \ldots, a_{k-1} \in \mathbb{R}$ such that

$$\mathbf{v}_k = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_{k-1}\mathbf{v}_{k-1} - \mathbf{v}_k = \mathbf{0}$$

□

Examples

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \right\}, \qquad \left\{ \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix} \right\}$$

$$a \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + b \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Longrightarrow a = b = 0$$

$\Longrightarrow$ linearly independent

$$2 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} - \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Longrightarrow$ not linearly independent (linearly dependent)

Summary

- Subspaces behave like ambient $\mathbb{R}^n$
- Set of actual outputs of a matrix are a subspace
- Useful to describe a subspace as a span of something
- Linear independence $\iff$ minimal