TMA4115 Matematikk 3

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Lecture 14: Determining Invertibility

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Key Points

- More tests for invertibility
- Murky Waters

Recap

- Invertible matrix: A is invertible if $\exists B : AB = I_m$ and $BA = I_n$
- Learnt so far:
  - $A \longrightarrow B$ via Row Reduction:
    A invertible $\iff$ B invertible
  - $A \longrightarrow I_m$ via Row Reduction
    $\implies A$ invertible
  - Invertible $\implies$ Square

The Result of Row Reduction

Full Gauss–Jordan Row Reduction

Example outcomes:

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & 8 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Neither is invertible

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$ impossible

Square Row Reduction Results

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & * \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The first is the only matrix such that:

1. A is invertible
   So A invertible $\iff$ row reduces to I
2. $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b}$
3. $A\mathbf{x} = \mathbf{b}$ has a unique solution for some $\mathbf{b}$
4. $A\mathbf{x} = \mathbf{0}$ has a unique solution

This is independent of size of square matrix

Story So Far

A is invertible

$\Longleftrightarrow$

A is square and one (hence all) of:

1. A row reduces to I_m
2. $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b}$
3. $A\mathbf{x} = \mathbf{b}$ has a unique solution for some $\mathbf{b}$
4. $A\mathbf{x} = \mathbf{0}$ has a unique solution

Most important is . . .

Computing The Inverse

The Amazing Gauss

GE is so amazing you can even find the inverse using it.

Idea

- A invertible $\iff$ A row reduces to I_m
- Row ops $\longleftrightarrow$ elementary matricces E_j:

$$E_k \cdots E_1 A = I_m$$

- Apply same row ops to I_m to get A^{-1}!

$$E_k \cdots E_1 I_m = E_k \cdots E_1 A A^{-1} = A^{-1}$$

- In practice: apply row ops to $[A|I_m]$.
  Result is $[I_m|B] \iff$ A invertible and $A^{-1} = B$

Example

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 3 \\ 1 & 4 & 6 \end{bmatrix} \qquad \text{Yes, } A^{-1} = \begin{bmatrix} 6 & -8 & 3 \\ -3 & 5 & -2 \\ 1 & -2 & 1 \end{bmatrix}$$

Is A invertible, if so what is A^{-1}?

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 3 & 3 & 0 & 1 & 0 \\ 1 & 4 & 6 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3-R_1]{R_2-R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 2 & 5 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3-2R_2]{R_1-2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 3 & -2 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right]$$

$$\xrightarrow[R_2-2R_3]{R_1+3R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 6 & -8 & 3 \\ 0 & 1 & 0 & -3 & 5 & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right]$$

Example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad \text{No, } A \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Is A invertible, if so what is A^{-1}?

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 7 & 8 & 9 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_2 - 4R_1 \\ \longrightarrow \\ R_3 - 7R_1 \end{array} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 0 & -6 & -12 & -7 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} R_3 - 2R_2 \\ \longrightarrow \end{array} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & -2\tfrac{2}{3} & 1\tfrac{2}{3} \\ 0 & 1 & 2 & 0 & 2\tfrac{1}{3} & -1\tfrac{1}{3} \\ 0 & 0 & 0 & 1 & -2 & 1 \end{array}\right]$$

Other Methods

When is

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

invertible?
Can solve $A\mathbf{x} = \mathbf{b}$ for all $\mathbf{b}$
Apply GE ($a \neq 0$):

$$\left[\begin{array}{cc|c} a & b & b_1 \\ c & d & b_2 \end{array}\right] \xrightarrow[a\neq 0]{R_2 - {}^{c}/_{a} R_1} \left[\begin{array}{cc|c} a & b & b_1 \\ 0 & d - {}^{cb}/_{a} & b_2 - {}^{c}/_{a} b_1 \end{array}\right]$$

$$\xrightarrow{R_2 \to aR_2} \left[\begin{array}{cc|c} a & b & b_1 \\ 0 & ad - cb & ab_2 - cb_1 \end{array}\right]$$

If $a \neq 0$ then A invertible $\iff$ $ad - bc \neq 0$
If $c \neq 0$ then swap so A invertible $\iff$ $cb - ad \neq 0$
If $a = c = 0$ then A not invertible and $cb - ad = 0$

It's that Strange Sense of Deja Vu

y_1, y_2 solutions of an ODE
linearly independent

$\Longleftrightarrow$

$y_1 y_2' - y_1' y_2 \neq 0$

A
invertible

$\Longleftrightarrow$

$ad - bc \neq 0$

Not a coincidence!

Determining Invertibility

Definition

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

the determinant of A *is*

$$\det(A) := \begin{vmatrix} a & b \\ c & d \end{vmatrix} := ad - bc$$

Lemma

$$A \text{ invertible} \iff \det(A) \neq 0$$

Bigger and . . . Better?

Sort of Definition

Square matrices of any size have determinants.

Exact definition not exactly . . . memorable:

$$\det(A) = \sum_{\sigma \in \Sigma_n} (-1)^{\sigma} a_{11\sigma} \cdots a_{nn\sigma}$$

Best: Learn rules for computation.

Don't Know? Ask Gauss

Key Properties for Computation

1. $\det(I_n) = 1$
2. $\det(AB) = \det(A)\det(B)$
3. Triangular matrices:

$$\begin{vmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 7 \end{vmatrix} = 2 \times 5 \times 7 = 70$$

4. Row Swaps:

$$\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = - \begin{vmatrix} 4 & 5 \\ 2 & 3 \end{vmatrix}$$

5. Row Multipliers:

$$\begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix} = 2 \begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix}$$

Note: only one row affected.

Don't Know? Ask Gauss

6. Row Adds:

$$\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 0 & -1 \end{vmatrix}$$

   No change!

7. Works for columns too.

Conclusion

Whenever you need to compute something, use GE!

Mnemonic

$$\begin{vmatrix} 0 & 1 \\ 2 & 2 \end{vmatrix} = -2$$

Example

$$\begin{vmatrix} 0 & 2 & 3 \\ 3 & 4 & 0 \\ 1 & 6 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 6 & 2 \\ 3 & 4 & 0 \\ 0 & 2 & 3 \end{vmatrix} \quad (R_1 \leftrightarrow R_3)$$

$$= -\begin{vmatrix} 1 & 6 & 2 \\ 0 & -14 & -6 \\ 0 & 2 & 3 \end{vmatrix} \quad (R_2 - 3R_1) \qquad = -2\begin{vmatrix} 1 & 3 & 2 \\ 0 & -7 & -6 \\ 0 & 1 & 3 \end{vmatrix} \quad (C_2 \to {}^1\!/_2 C_2)$$

$$= 2\begin{vmatrix} 1 & 3 & 2 \\ 0 & 1 & 3 \\ 0 & -7 & -6 \end{vmatrix} \quad (R_2 \leftrightarrow R_3) \qquad = 2\begin{vmatrix} 1 & 3 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 15 \end{vmatrix} \quad (R_3 + 7R_2)$$

$$= 2 \times 15 = 30$$

Story So Far

- To compute anything:
  Use Gaussian Elimination

Muddiest Point

The Winner of the Muddiest Point Competition is

Inhomogeneous ODEs

Recall

$$y'' + p(x)y' + q(x)y = r(x)$$

Examples

1. $y'' + y' - 2y = \sin(x)$
2. $x^2y'' + xy' - y = x^2 + x^3$
3. $y'' + x^{-2}y' - x^{-3}y = x^{-3}$

General Method

1. Solve homogeneous ODE, y_1, y_2
   1.1 "Well-known ODE" $\Longrightarrow$ "look up" solution
   1.2 Otherwise, guess one solution
   1.3 "Reduction of order" for other
2. Find a solution of inhomogeneous ODE, y_p
   2.1 Guess
   2.2 Otherwise, use "variation of parameters"
3. Fit to initial values:

$$y_p(x_0) + a y_1(x_0) + b y_2(x_0)$$
$$y_p'(x_0) + a y_1'(x_0) + b y_2'(x_0)$$

Examples

$$y'' + y' - 2y = \sin(x)$$

Examples

$$x^2 y'' + xy' - y = x^2 + x^3$$

Examples

$$y'' + x^{-2}y' - x^{-3}y = x^{-3}$$

Summary

- Matrices:
  Use GE whenever you want an actual answer
- ODEs:
  Use intelligent guesswork