TMA4115 Matematikk 3

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Lecture 10: Elementary, my dear Gauss

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Recap

- Linear systems: things like

$$-x + 4y = 9$$
$$3x + 2y = 1$$

- Solve: means find numbers that make them true
- Some are easier to solve than others

Key Points

- Elementary operations change linear systems without changing solutions
- One other . . . to be revealed

Which Is Easier?

1. Solve

$$-x + 4y = 9$$
$$3x + 2y = 1$$

2. Solve

$$-x + 8 = 9$$
$$y = 2$$

Key Idea

Solutions are same so transform first into second

Secondary Goal

Transform:
linear systems of 2nd type into linear systems of 1st type.

Simple Steps: Elementary Operations

1. Scaling: $\lambda \neq 0$

$$
\begin{aligned}
a_1 x_1 + \quad a_2 x_2 + \ldots + \quad a_n x_n &= \quad b \\
&\downarrow \\
\lambda a_1 x_1 + \lambda a_2 x_2 + \ldots + \lambda a_n x_n &= \lambda b
\end{aligned}
$$

2. Swapping:

$$
\begin{aligned}
a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n &= b_1 \\
a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n &= b_2 \\
&\downarrow \\
a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n &= b_2 \\
a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n &= b_1
\end{aligned}
$$

Simple Steps: Elementary Operations

3. Adding: any $\lambda \in \mathbb{R}$

$$
\begin{array}{rcl}
a_{11}x_1 + & a_{12}x_2 + \ldots + & a_{1n}x_n = b_1 \\
a_{21}x_1 + & a_{22}x_2 + \ldots + & a_{2n}x_n = b_2 \\
 & \downarrow & \\
(a_{11}+\lambda a_{21})x_1 + & (a_{12}+\lambda a_{22})x_2 + \ldots + & (a_{1n}+\lambda a_{2n})x_n = b_1+\lambda b_2 \\
a_{21}x_1 + & a_{22}x_2 + \ldots + & a_{2n}x_n = b_2
\end{array}
$$

Finding Solutions

What Makes It Work

In each case: if $x_1, x_2, \ldots, x_n$ satisfy 1st set, they also satisfy 2nd set.

1. Scaling: $\lambda \neq 0$
   if $x_1, x_2, \ldots, x_n$ are such that

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

   then $x_1, x_2, \ldots, x_n$ are such that

$$\begin{aligned} \lambda a_1x_1 + \lambda a_2x_2 + \cdots + \lambda a_nx_n & \\ &= \lambda(a_1x_1 + a_2x_2 + \cdots + a_nx_n) \\ &= \lambda b \end{aligned}$$

Finding Solutions

2. Swapping:
   if $x_1, x_2, \ldots, x_n$ are such that

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$

   then $x_1, x_2, \ldots, x_n$ are such that

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$
$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$

Finding Solutions

3. Adding: any $\lambda \in \mathbb{R}$
   if $x_1, x_2, \ldots, x_n$ are such that

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \end{aligned}$$

   then $x_1, x_2, \ldots, x_n$ are such that

$$\begin{aligned} (a_{11}+\lambda a_{21})x_1 + (a_{12}+\lambda a_{22})x_2 + \ldots + (a_{1n}+\lambda a_{2n})x_n &= b_1+\lambda b_2 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \end{aligned}$$

“If” versus “Iff”

Solution of 1st set is solution of 2nd.

Also want reverse: solution of 2nd is solution of 1st.

Comes from reversibility:

1. Scaling: scale by λ^{-1}
2. Swapping: swap back
3. Adding: add $-\lambda$ of 2nd to 1st

Examples

$$\begin{aligned} 2x + 3y + z &= -2 \\ 2x - 3y + 4z &= 1 \\ -x + 2y - z &= -2 \end{aligned} \xrightarrow{R_2 - R_1} \begin{aligned} 2x + 3y + z &= -2 \\ -6y + 3z &= 3 \\ -x + 2y - z &= -2 \end{aligned}$$

$$\xrightarrow{R_3 + {}^1\!/_2 R_1} \begin{aligned} 2x + 3y + z &= -2 \\ -6y + 3z &= 3 \\ 3{}^1\!/_2 y - {}^1\!/_2 z &= -3 \end{aligned}$$

$$\xrightarrow{R_3 + {}^7\!/_{12} R_2} \begin{aligned} 2x + 3y + z &= -2 \\ -6y + 3z &= 3 \\ 1{}^1\!/_4 z &= -1{}^1\!/_4 \end{aligned}$$

Stop here. Read off solution: $z = -1$, so $y = -1$, so $x = 1$

Slightly Simpler

$$\begin{aligned} 2x + 3y + z &= -2 \\ 2x - 3y + 4z &= 1 \\ -x + 2y - z &= -2 \end{aligned} \xrightarrow{R_1 \leftrightarrow R_3} \begin{aligned} -x + 2y - z &= -2 \\ 2x - 3y + 4z &= 1 \\ 2x + 3y + z &= -2 \end{aligned}$$

$$\xrightarrow[R_3+2R_1]{R_2+2R_1} \begin{aligned} -x + 2y - z &= -2 \\ y + 2z &= -3 \\ 7y - z &= -6 \end{aligned}$$

$$\xrightarrow{R_3+7R_2} \begin{aligned} -x + 2y - z &= -2 \\ y + 2z &= -3 \\ -15z &= 15 \end{aligned}$$

Difference? No nasty fractions!

The Key Step

1. Scaling
2. Swapping
3. Adding

Reason

It's the only one that substantially alters the system.

Project

Design a “program” to solve linear systems.

Things to focus on:

- Specifying the problem
- Designing the algorithm
- Implementing the algorithm
- Interpreting the result
- Important Technicalities
  - Inputting the problem
  - Ensuring termination

This frame has been deliberately left blank.

Matrix

Definition

A matrix is a rectangular grid of things.

Assume (real) numbers.

Key Point

Linear systems can be represented by matrices
with no loss of information

The Obligatory Matrix Stunt

$$\begin{aligned} 2x + \quad 3y + \quad z &= -2 \\ 2x - \quad 3y + \quad 4z &= 1 \\ -x + \quad 2y - \quad z &= -2 \end{aligned}$$

The Rules

- Keep only the numbers
- Include "hidden" 0s, 1s, and -1s:

$$\text{“ ”} = 0x, \qquad x = 1x, \qquad -x = -1x$$

- Keep watch for signs

Two Matrices

- Coefficient matrix: left-hand side of "="s
- Augmented matrix: everything

Why? Often vary right-hand side but keep left-hand side same.

Notation:

$$\begin{bmatrix} 2 & 3 & 0 \\ 3 & -3 & 4 \\ -1 & 2 & -1 \end{bmatrix} \qquad \left[\begin{array}{ccc|c} 2 & 3 & 0 & -2 \\ 2 & -3 & 4 & 1 \\ -1 & 2 & -1 & -2 \end{array}\right]$$

Summary

- Use Gaussian elimination to solve linear systems
- Result is in echelon form
- Reduced echelon form is guaranteed unique
- Matrices encode linear systems
- Can perform GE on matrix rather than linear system