TMA4115 Matematikk 3

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Lecture 9: What's The Difference?

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Key Points

- Focus on solving problem
- Find useful description of solution set
- Use reversible techniques

Linear Systems

Linear $\Longrightarrow$ constants + multiples of x, y, z (etc.)

Linear $\Longrightarrow$ no funny functions!

$$\begin{aligned} 2x + \quad & 3y + \quad & 4z = 7 \quad & \checkmark \\ 2x^2 + \quad & 3\sqrt{y} + & 4\sin(z) = 7 \quad & \times \\ \sin(2)x + & \log(3)y + \quad & e^4 z = 7 \quad & \checkmark \end{aligned}$$

Linear system = family of linear equations.

Key Question

What do we do with them?

Answer

Solve them!

Solving Linear Systems

Solve

$$-x + 4y = 9$$
$$3x + 2y = 1$$

Question

What does that mean?

Answer

- Find two numbers which when put in for x and y make the two statements true.
- Find all possible such pairs, or show that none exist.

ODEs and Linear Systems

Similarities

- Focus on solving

$\Longrightarrow$ Method subordinate to goal

- In practise, use approximation methods

Differences

- Theoretically possible to solve exactly

Since You Mentioned Them. . .

Linear Systems From ODEs

The final stage in solving an IVP is a linear system

Linear Systems from ODEs

Solve

$$y'' + 2y' + y = 0, \qquad y(0) = 1, y'(0) = 0$$

- Solutions: e^{-x}, xe^{-x}
- General Solution: $ae^{-x} + bxe^{-x}$
- Fit to IVP:

$$ae^{-0} + b0e^{-0} = 1$$
$$-ae^{-0} - b0e^{-0} + be^{-0} = 0$$

- Rearrange:

$$a + 0b = 1$$
$$-a + b = 0$$

- Read off solution: $a = 1$, $b = 1$
- Interpret correctly: solution of IVP is $e^{-x} + xe^{-x}$

Solving Linear Systems: Key Step

Solution of

$$\begin{aligned} -x + 4y &= 9 \\ 3x + 2y &= 1 \end{aligned}$$

is solution of
and vice versa

$$\begin{aligned} -x + 4y &= 9 \\ 14y &= 28 \end{aligned}$$

is solution of
and vice versa

$$\begin{aligned} -x + 4y &= 9 \\ y &= 2 \end{aligned}$$

is solution of
and vice versa

$$\begin{aligned} -x + 8 &= 9 \\ y &= 2 \end{aligned}$$

is solution of
and vice versa

$$\begin{aligned} x &= -1 \\ y &= 2 \end{aligned}$$

Which Is Easier?

1. Solve

$$-x + 4y = 9$$
$$3x + 2y = 1$$

2. Solve

$$-x + 8 = 9$$
$$y = 2$$

Key Idea

Solutions are same so transform first into second

Secondary Goal

Transform:
linear systems of 2nd type into linear systems of 1st type.

Simple Steps: Elementary Operations

1. Scaling: $\lambda \neq 0$

$$
\begin{aligned}
a_1 x_1 + \quad a_2 x_2 + \ldots + \quad a_n x_n &= \quad b \\
&\downarrow \\
\lambda a_1 x_1 + \lambda a_2 x_2 + \ldots + \lambda a_n x_n &= \lambda b
\end{aligned}
$$

2. Swapping:

$$
\begin{aligned}
a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n &= b_1 \\
a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n &= b_2 \\
&\downarrow \\
a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n &= b_2 \\
a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n &= b_1
\end{aligned}
$$

Simple Steps: Elementary Operations

3. Adding: any $\lambda \in \mathbb{R}$

$$
\begin{array}{rcl}
a_{11}x_1 + & a_{12}x_2 + \ldots + & a_{1n}x_n = b_1 \\
a_{21}x_1 + & a_{22}x_2 + \ldots + & a_{2n}x_n = b_2 \\
 & \downarrow & \\
(a_{11}+\lambda a_{21})x_1 + & (a_{12}+\lambda a_{22})x_2 + \ldots + & (a_{1n}+\lambda a_{2n})x_n = b_1+\lambda b_2 \\
a_{21}x_1 + & a_{22}x_2 + \ldots + & a_{2n}x_n = b_2
\end{array}
$$

Finding Solutions

- In each case: if $x_1, x_2, \ldots, x_n$ satisfy 1st set, they also satisfy 2nd set.
- Scaling: $c = d \Longrightarrow \lambda c = \lambda d$
- Swapping: $\big(a = b \text{ and } c = d\big) \Longrightarrow \big(c = d \text{ and } a = b\big)$
- Adding:
  $\big(a = b \text{ and } c = d\big) \Longrightarrow \big(a + \lambda c = b + \lambda d \text{ and } c = d\big)$

$\Longrightarrow$ in each case, solution of 1st set is solution of 2nd.

"$\Longleftrightarrow$" comes from reversibility

Each step can be reversed by a similar step.

Examples

$$\begin{array}{rrrr} 2x+ & 3y+ & z= & -2 \\ 2x- & 3y+ & 4z= & 1 \\ -x+ & 2y- & z= & -2 \end{array} \xrightarrow{R_2-R_1} \begin{array}{rrrr} 2x+ & 3y+ & z= & -2 \\ & -6y+ & 3z= & 3 \\ -x+ & 2y- & z= & -2 \end{array}$$

$$\xrightarrow{R_3+{}^1\!/_2 R_1} \begin{array}{rrrr} 2x+ & 3y+ & z= & -2 \\ & -6y+ & 3z= & 3 \\ & 3{}^1\!/_2 y- & {}^1\!/_2 z= & -3 \end{array}$$

$$\xrightarrow{R_3+{}^7\!/_{12} R_2} \begin{array}{rrrr} 2x+ & 3y+ & z= & -2 \\ & -6y+ & 3z= & 3 \\ & & 1{}^1\!/_4 z= & -1{}^1\!/_4 \end{array}$$

Stop here. Read off solution: $z=-1$, so $y=-1$, so $x=1$

Slightly Simpler

$$\begin{array}{rcl} 2x+3y+z=-2 & & -x+2y-z=-2 \\ 2x-3y+4z=1 & \xrightarrow{R_1\leftrightarrow R_3} & 2x-3y+4z=1 \\ -x+2y-z=-2 & \xrightarrow{R_3+2R_1} & 2x+3y+z=-2 \end{array}$$

$$\begin{array}{rcl} & & -x+2y-z=-2 \\ & \xrightarrow{R_2+2R_1} & y+2z=-3 \\ & \xrightarrow{R_3+2R_1} & 7y-z=-6 \end{array}$$

$$\begin{array}{rcl} & & -x+2y-z=-2 \\ & \xrightarrow{R_3+7R_2} & y+2z=-3 \\ & & -15z=15 \end{array}$$

Difference? No nasty fractions!

Summary

- Solving linear systems means finding numbers that make them true
- Some linear systems are easier than others
- Elementary operations change linear systems without changing solutions
- Choose to make simpler