TMA4115 Matematikk 3

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Lecture 8: Ode to ODE

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Recap

- Solve inhomogeneous by:
  1. Solve homogeneous
  2. Find particular solution
  3. Solve IVP
- Constant coefficients + periodic forcing displays . . . unusual behaviour

Key Points

- Use solutions of homogeneous to “simplify” search for solution of inhomogeneous

Lessons From Pendula

Mathematical Models are

1. Idealised
2. Simplified
3. Limited validity

Strange Behaviour

1. Error in Implementation

   Fix it, and hope no-one notices.

2. Error in Model

   Find boundaries in model.

3. Error in Reality

   Publish and get a Nobel prize!

Form(ula) versus Function

Definition

A function $f : I \to \mathbb{R}$ *is a*

rule

assigning a real number to each number in I*.*

Example

1. $t \mapsto t^2$
2. $t \mapsto \sin(t)$
3. $\theta \mapsto x$–coordinate of point with angle θ
4. $t \mapsto t$–value of solution of

$$y'' + p(x)y' + q(x)y = r(x)$$

Unknown Knowns

Key Point

- $f(t)$ must be uniquely determined
- $f(t)$ need not be known

Known versus Determined

Value of solution of

$$y'' + (y')^2 + 1 = 0$$

at $x = \pi/2$?

$$0$$

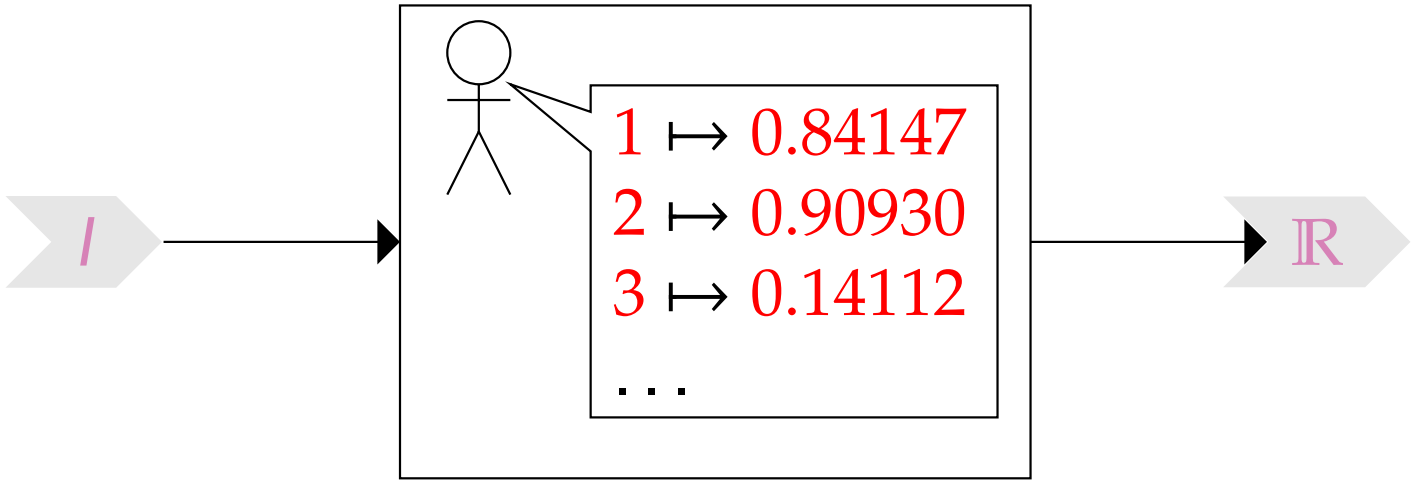
Impersonal Definition

- Function does not depend on knowledge of user
- Function does not depend on exact implementation

Example

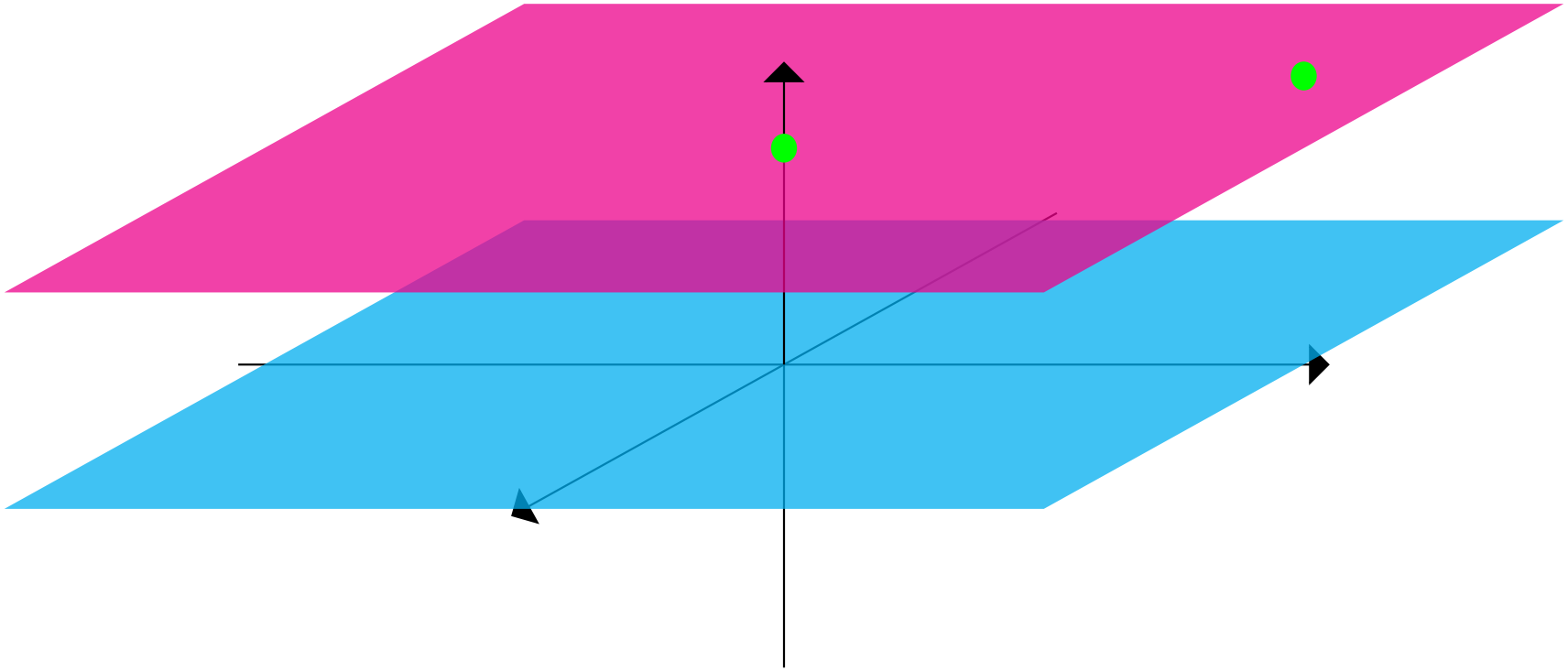
1. "$t \mapsto t$–value of solution of ODE" is fine
2. $2\sin(x)\cos(x)$ and $\sin(2x)$ are same function

Form(ula) versus Function



III : Implementation Is Irrelevant

Finding a Plane: the ODE Way



1. Find plane through origin.
2. Find point on plane.
3. Simplify search: look in “obvious” direction.

Spot of Geometry

- Plane through origin: $\{\lambda\vec{v} + \mu\vec{w}\}$
- Translated plane: $\{\vec{u} + \lambda\vec{v} + \mu\vec{w}\}$

- Lots of choices for $\vec{u}$.
- "Best" choice has $\vec{u}$ orthogonal to plane:

$$\vec{u}^{\mathsf{T}}\vec{v} = 0, \qquad \vec{u}^{\mathsf{T}}\vec{w} = 0$$

- Makes problem simpler by artifically imposing additional constraints.
- $\Big($ General nonsense $\Longrightarrow$ no loss of generality $\Big)$

Simplifying Problems

$$y'' + y = \tan\left(\log\left(\frac{\sin(x)}{2+\cos(x)}\right)\right)$$

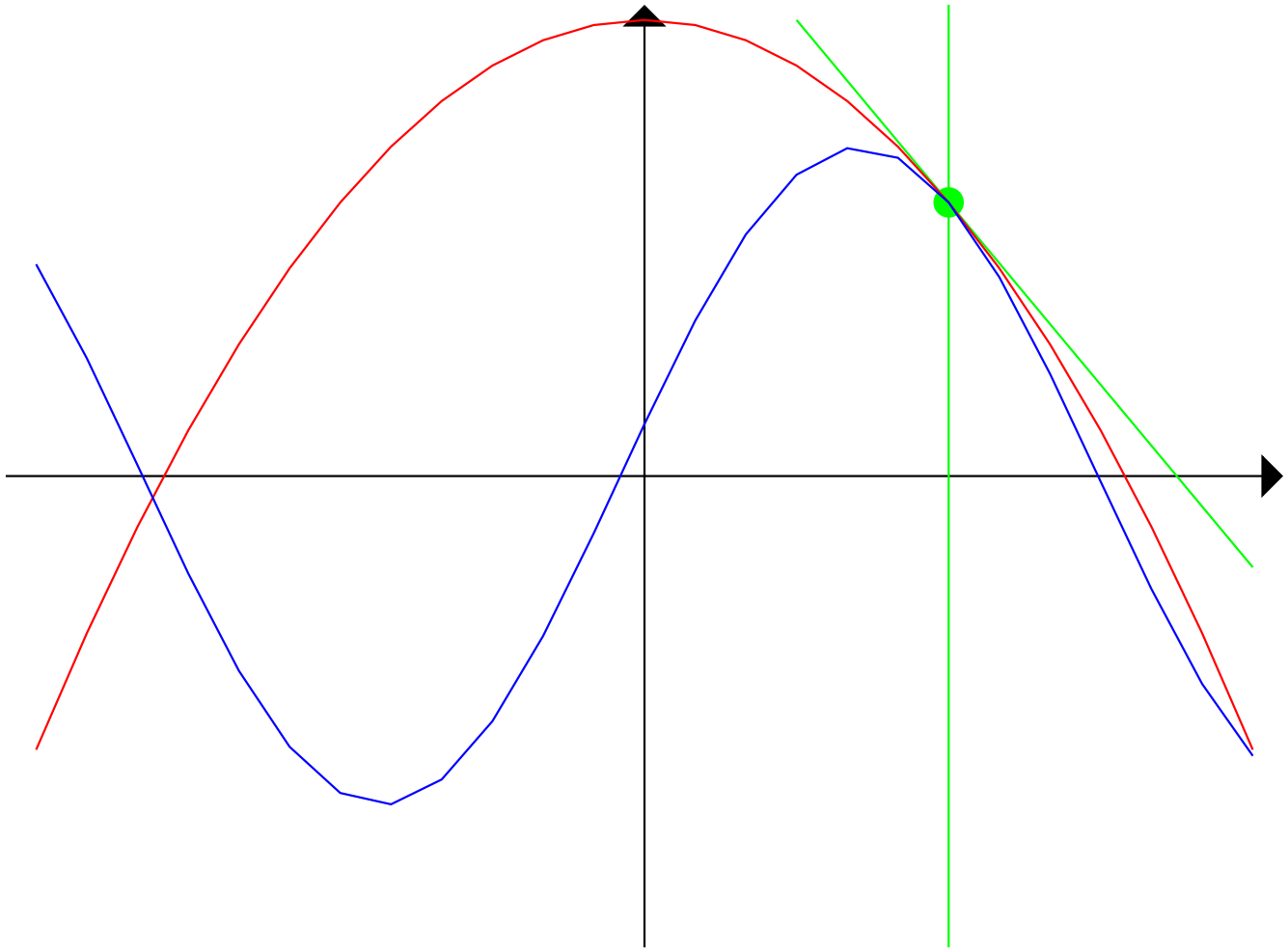
Solutions to Homogeneous: $\sin(x)$, $\cos(x)$.

Solution of Inhomogeneous? $y(x)$?

- $y(x)$ fixed by $y(x_0)$, $y'(x_0)$, any $x_0 \in I$.
- Have solution of homogeneous ODE through the same points:

$$c_{x_0}\sin(x) + d_{x_0}\cos(x)$$

In Pictures



Known Unknowns

Key Principle

Approximate unknown by most useful knowns.

- In this case, solutions of homogeneous ODE.
- Computation: polynomials
- Signal analysis: trigonomials
- Plotting: piecewise-linear
- ...

What's the Point?

$$y(x) = c_{x_0} \sin(x) + d_{x_0} \cos(x)$$

- only 100% accurate at x_0 could be bad elsewhere!
- but for each x_0 there's a unique (c_{x_0}, d_{x_0}) that works
- $x_0 \mapsto c_{x_0}$ and $x_0 \mapsto d_{x_0}$ are functions!

Conclusion

Have functions $c(x)$, $d(x)$ such that for all $x \in I$

$$y(x) = c(x)y_1(x) + d(x)y_2(x)$$
$$y'(x) = c(x)y_1'(x) + d(x)y_2'(x)$$

So What?

Know $y(x)$

$\Longleftrightarrow$

Know $c(x)$ and $d(x)$

Bad Trade?

1 unknown function $\longleftrightarrow$ 2 unknown functions

Maybe we gain elsewhere...

The Trade

Conditions:

$$y(x) = c(x)y_1(x) + d(x)y_2(x) \quad (1)$$
$$y'(x) = c(x)y_1'(x) + d(x)y_2'(x) \quad (2)$$

- But (2) is not the derivative of (1)!

Differentiating:

$$y'(x) = c(x)y_1'(x) + d(x)y_2'(x) + c'(x)y_1(x) + d'(x)y_2(x)$$

Thus:

$$0 = c'(x)y_1(x) + d'(x)y_2(x)$$

Use The ODE, Luke

$$y'(x) = c(x)y_1'(x) + d(x)y_2'(x)$$

Differentiating Again:

$$y''(x) = c(x)y_1''(x) + d(x)y_2''(x) + c'(x)y_1'(x) + d'(x)y_2'(x)$$

Substituting In:

$$\begin{aligned} y'' + p(x)y' + q(x)y &= r(x) \\ c(x)y_1''(x) + d(x)y_2''(x) + c'(x)y_1'(x) + d'(x)y_2'(x) & \\ +p(x)\Big(c(x)y_1'(x) + d(x)y_2'(x)\Big) & \\ +q(x)\Big(c(x)y_1(x) + d(x)y_2(x)\Big) &= r(x) \end{aligned}$$

Rearranging

$$
\begin{aligned}
y'' + p(x)y' + q(x)y &= r(x) \\
cy_1'' + dy_2'' + c'y_1' + d'y_2' & \\
+p\left(cy_1' + dy_2'\right) & \\
+q\left(cy_1 + dy_2\right) &= r \\
c\left(y_1'' + py_1' + qy_1\right) & \\
+ d\left(y_2'' + py_2' + qy_2\right) & \\
c'y_1' + d'y_2' &= r \\
c'y_1 + d'y_2 &= 0
\end{aligned}
$$

Solving

$$\begin{aligned} c'y_1'y_1 + d'y_2'y_1 &= ry_1 \\ c'y_1y_1' + d'y_2y_1' &= 0 \end{aligned}$$

$$d'\left(y_1y_2' - y_1'y_2\right) = ry_1$$

$$d = \int \frac{ry_1}{W(y_1, y_2)}\mathrm{d}x$$

$$c = -\int \frac{ry_2}{W(y_1, y_2)}\mathrm{d}x$$

Variation of Parameters

Particular Solution

$$y(x) = c(x)y_1(x) + d(x)y_2(x)$$

$$c(x) = -\int^x \frac{r(t)y_2(t)}{W(y_1, y_2)(t)}\mathrm{d}t$$

$$d(x) = \int^x \frac{r(t)y_1(t)}{W(y_1, y_2)(t)}\mathrm{d}t$$

Example

$$y'' - 2y' + y = e^x \sin(x)$$

Homogeneous: e^x, xe^x
Wronskian: $W(y_1, y_2) = e^x(e^x + xe^x) - e^x xe^x = e^{2x}$
Variation of Parameters:

$$\begin{aligned} c(x) &= -\int^x \frac{e^t \sin(t) t e^t}{e^{2t}} dt \\ &= -\int^x t \sin(t) dt \\ &= [t \cos(t)]^x - \int^x \cos(t) dt \\ &= x \cos(x) - \sin(x) \end{aligned}$$

Example ctd.

$$
\begin{aligned}
d(x) &= \int^{x} \frac{e^{t}\sin(t)e^{t}}{e^{2t}}\,dt \\
&= \int^{x} \sin(t)\,dt \\
&= -\cos(x)
\end{aligned}
$$

General Solution:

$$
\Big(x\cos(x) - \sin(x)\Big)e^{x} - \cos(x)xe^{x} + ae^{x} + bxe^{x}
$$

$$
= -\sin(x)e^{x} + ae^{x} + bxe^{x}
$$

Always check!

Example

$$x^2 y'' - 2x^{-1} y' + 2y = x^3 \cos(x)$$

Homogeneous: x, x^2
Wronskian: $W(y_1, y_2) = x2x - x^2 = x^2$
Variation of Parameters:

$$\begin{aligned} c(x) &= -\int^x \frac{t\cos(t)t^2}{t^2}\mathrm{d}t \\ &= -\int^x t\cos(t)\mathrm{d}t \\ &= -x\sin(x) - \cos(x) \end{aligned}$$

Example ctd

$$
\begin{aligned}
d(x) &= \int^{x} \frac{t\cos(t)t}{t^2}\mathrm{d}t \\
&= \int^{x} \cos(t)\mathrm{d}t \\
&= \sin(x)
\end{aligned}
$$

General Solution:

$$
\begin{aligned}
-\Big(x\sin(x)+\cos(x)\Big)x + x^2\sin(x) + ax + bx^2 \\
= -x\cos(x) + ax + bx^2
\end{aligned}
$$

Summary

- ODEs have solutions
- No guaranteed method for finding
- Techniques for special cases
- Use with care!
- Use partial information to get more