TMA4115 Matematikk 3

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Lecture 4: Put a Spring in Your ODE

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Key Points

- Reduction of Order: An Example
- Solving Linear ODEs with Constant Coefficients

Detailed Recap

- 2nd Order Linear Homogeneous ODE:

$$y'' + p(x)y' + q(x)y = 0 \tag{1}$$

- Initial conditions:

$$y(0) = a, \qquad y'(0) = b \tag{2}$$

- Problem: Solve it
- Solution:
  1. Find 2 functions, y_1, y_2, such that 1 holds for both.
  2. Show linear indepdenence
  3. Solve

$$cy_1(0) + dy_2(0) = a$$
$$cy_1'(0) + dy_2'(0) = b$$

Reduction of Order: Example

$$(x^2-2x)y''-(x^2-2)y'+2(x-1)y=0, \quad y(1)=1,\ y'(1)=0.$$

Note

- Not in standard form.
- Initial conditions not “initial” (not at 0)!
  Not a problem!
  “Initial conditions” means “specified at one point”.
  (Avoid 0 here as ODE does not make sense there.)

Reduction of Order: Example

$$y'' - \frac{x^2-2}{x^2-2x}y' + \frac{2(x-1)}{x^2-2x}y = 0$$

- Find (guess!) a solution.
- Start by putting in simple functions.
- Polynomials suggest: try monomials, x^k

$$0 = k(k-1)x^{k-2} - k\frac{x^2-2}{x^2-2x}x^{k-1} + \frac{2(x-1)}{x^2-2x}x^k$$

$$\Rightarrow 0 = k(k-1)(x^k - 2x^{k-1}) - k(x^2-2)x^{k-1} + 2(x-1)x^k$$

$$\Rightarrow 0 = (2-k)x^{k+1} + (k(k-1)-2)x^k + 2k(k-2)x^{k-1}$$

- So x^2 works.

Reduction of Order: Example

$$y'' - \frac{x^2 - 2}{x^2 - 2x} y' + \frac{2(x-1)}{x^2 - 2x} y = 0$$

$$\begin{aligned} y &= ux^2 \\ y' &= u'x^2 + 2ux \\ y'' &= u''x^2 + 4u'x + 2u \end{aligned}$$

$$\begin{aligned} 0 &= y'' - \frac{x^2 - 2}{x^2 - 2x} y' + \frac{2(x-1)}{x^2 - 2x} y \\ &= u''x^2 + 4u'x + 2u \\ &\quad - \frac{x^2 - 2}{x^2 - 2x}(u'x^2 + 2ux) + \frac{2(x-1)}{x^2 - 2x} ux^2 \end{aligned}$$

Reduction of Order: Example

$$0 = u''x^2 + 4u'x + 2u$$
$$- \frac{x^2 - 2}{x^2 - 2x}(u'x^2 + 2ux) + \frac{2(x-1)}{x^2 - 2x}ux^2$$

Fast Track: Only terms in u'' and u' will "survive".

$$0 = u''x^2 + \left(4x^{-1} - 1 - \frac{x^2(x^2-2)}{x^2 - 2x}\right)u'$$

Note obvious solution: $u = \text{const}$.
Gives x^2 as solution of original ODE — which we already knew! So ignore.

Reduction of Order: Example

$$u'' + \left(4x^{-1} - 1 - \frac{2x-2}{x^2-2x}\right)u' = 0$$

$$U' + \left(4x^{-1} - 1 - \frac{2x-2}{x^2-2x}\right)U = 0$$

$$\int U' \mathrm{d}x = \int \left(-4x^{-1} + 1 + \frac{2x-2}{x^2-2x}\right) U \mathrm{d}x$$

$$\ln(U) = -4\ln(x) + x + \ln(x^2-2x)$$

$$u = \int U \mathrm{d}x = \int x^{-4}(x^{-2} - 2x^{-3})\mathrm{e}^x \mathrm{d}x = x^{-2}\mathrm{e}^x$$

$$y = x^2 u = \mathrm{e}^x$$

Reduction of Order: Example

$$(x^2 - 2x)y'' - (x^2 - 2)y' + 2(x - 1)y = 0$$

has solutions x^2, e^x
(Ought to check . . . exercise!)
Initial conditions: $y(1) = 1$, $y'(1) = 0$
Substituting in:

$$y(x) = ax^2 + be^x$$
$$y'(x) = 2ax + be^x$$
$$1 = a + be$$
$$0 = 2a + be$$

Obtain: $y(x) = -x^2 + 2e^{x+1}$

Reduction of Order: Conclusion

- Starts with a guess, y_1
- After that, algorithmic (though can be complicated!)
- Key step: substitute $y = uy_1$ and solve for u'

Simple Harmonic Motion

Introducing . . .

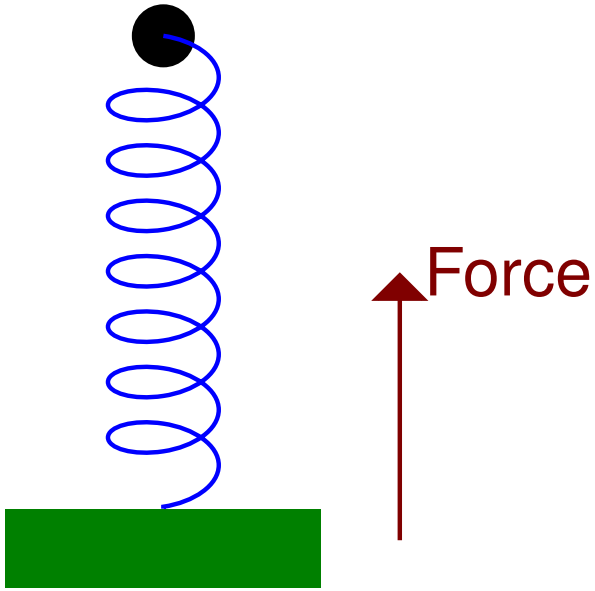
A very special ODE:

$$y'' + ay' + by = 0$$

2nd Order Linear Homogeneous ODE
with constant coefficients

Occurs in models of many physical systems

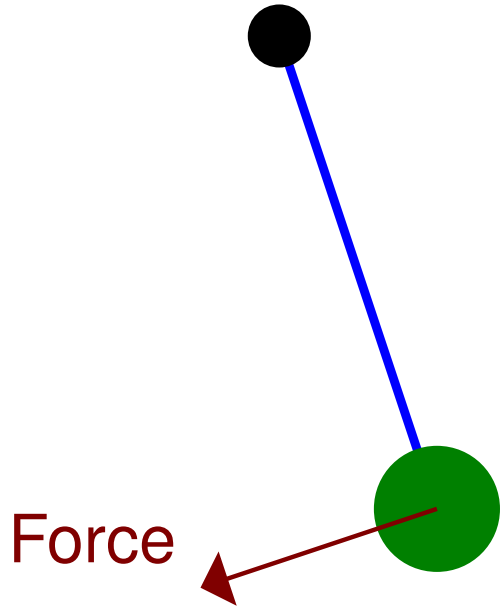
Spring Dynamics



- Newton: $F_{\text{total}} = mx''$
- Hooke: $F_{\text{spring}} = -x$
- Friction: $F_{\text{friction}} = -cx'$
- In total:

$$mx'' + cx' + kx = 0$$

Pendular Dynamics



- Newton: $F_{\text{total}} = -mx'' = ml\theta''$ ($x =$ arc length)
- $F_{\text{gravity}} = mg\sin(\theta) = mg\theta$
  Too complicated! Write θ instead.
- Friction: as before
- In total:

$$l\theta'' + c\theta' + g\theta = 0$$

Solutions

$$y'' + ay' + by = 0$$

1st Order Variant:

$$y' + \lambda y = 0$$

Solution: $y(x) = ce^{\lambda x}$
See if it works for 2nd Order:

$$\lambda^2 e^{\lambda x} + a\lambda e^{\lambda x} + be^{\lambda x} = 0$$

True if and only if

$$\lambda^2 + a\lambda + b = 0$$

Solutions: Grand Claim

$$y'' + ay' + by = 0$$

has solutions

$$e^{\lambda_1 x}, e^{\lambda_2 x}$$

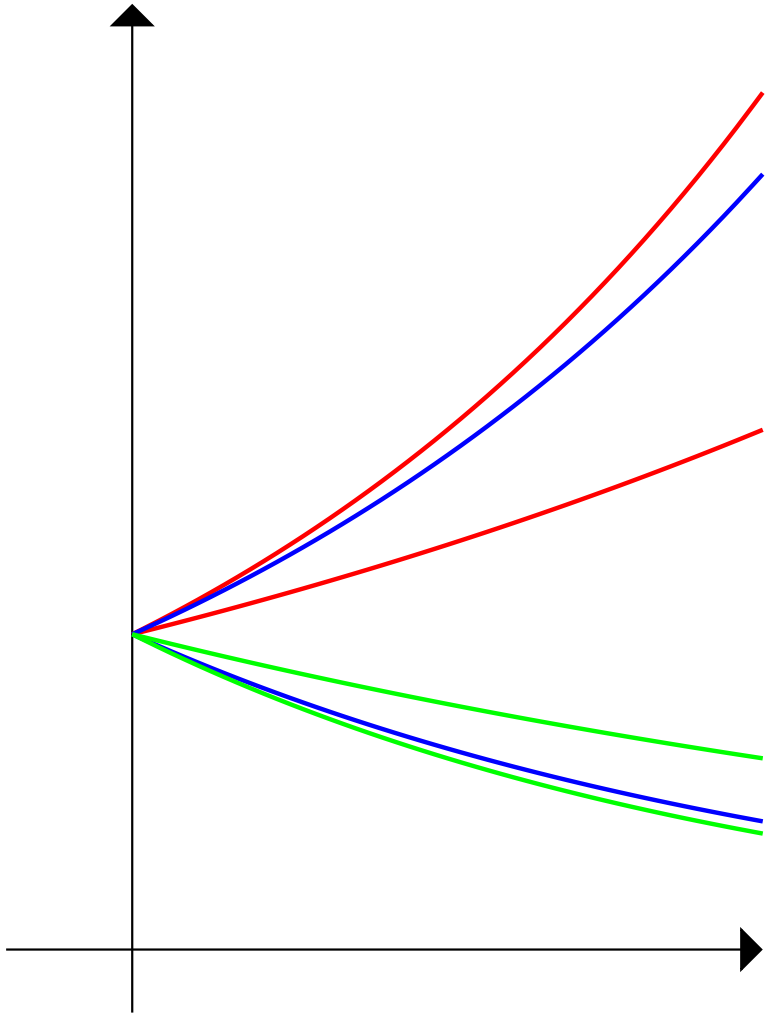
where λ_1, λ_2 roots of

$$\lambda^2 + a\lambda + b$$

Two solutions, so done sort of.

Complications

1. $a^2 - 4b > 0$: 2 real roots, no problems.

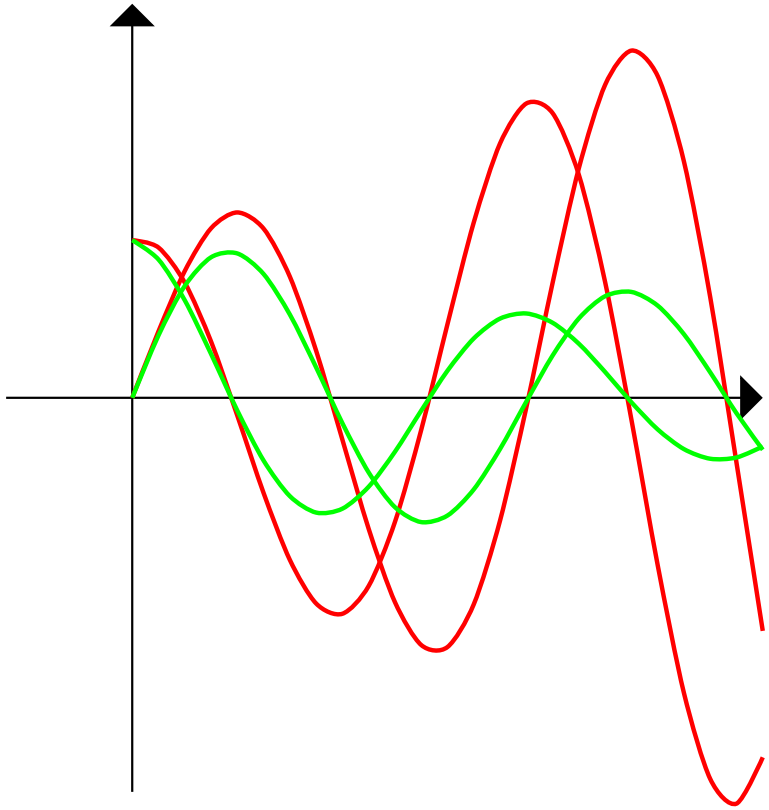


Complications

2. $a^2 - 4b < 0$: 2 complex roots, no real problems.

Just remember $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ to get

$$e^{\lambda x}\cos(\pm\omega x + \theta_0)$$



Complications

3. $a^2 - 4b = 0$: 1 root! ($\lambda = -a/2$) Problems!

Only have one solution so far.
Use "Reduction" to find other: $y = ue^{\lambda x}$.
Get:

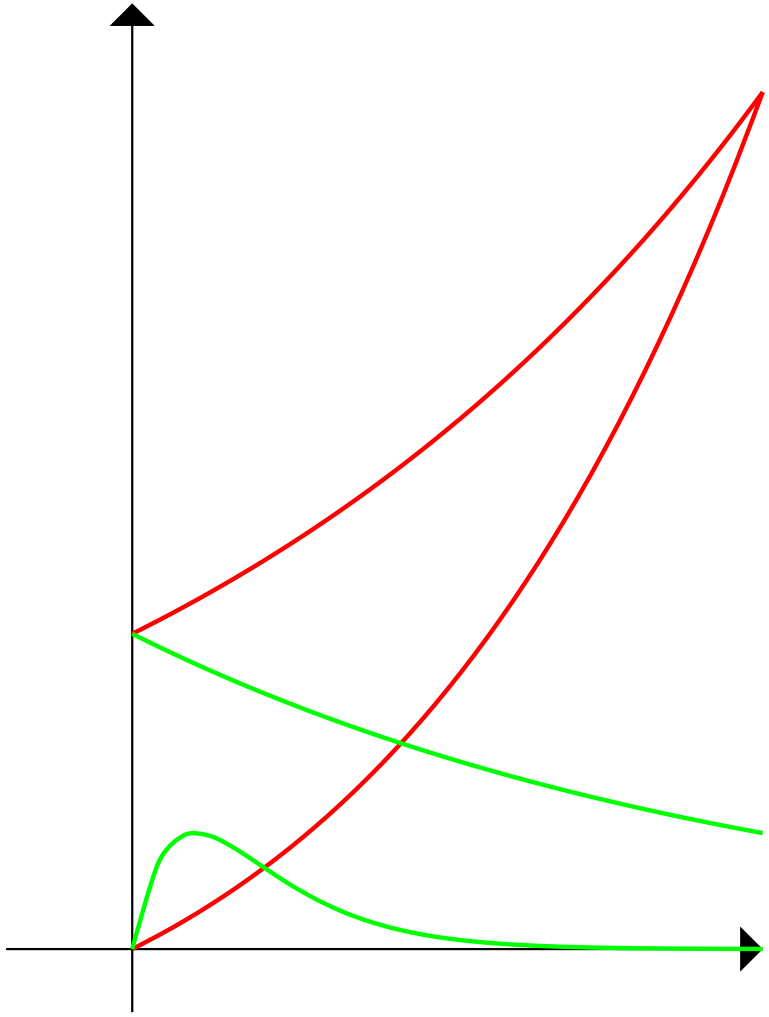
$$\left(u'' + (2\lambda + a)u' + (\lambda^2 + a\lambda + b)u\right)e^{\lambda x}$$

which simplifies to:

$$u'' = 0$$

Solution: $u = x$, hence $xe^{\lambda x}$ is the other solution.

Repeated Roots



Summary

- Solving ODEs means Finding Solutions by any means
- Can use knowledge about ODE to help solve it
- Constant Coefficients has solutions

$$e^{\lambda_1 x}, \quad e^{\lambda_2 x}$$

$$e^{\lambda x}\cos(\omega x + \theta_0), \quad e^{\lambda x}\cos(-\omega x + \theta_0)$$

$$e^{\lambda x}, \quad x e^{\lambda x}$$