TMA4115 Matematikk 3

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Lecture 1: y i?

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Introducing . . .

$$\sqrt{-1}$$

Usual Questions

Questions

What?	Square root of -1
Why?	Because it's useful
How?	Ah, now that's an interesting question . . .

More detail on “why”

Mathematics is a language:

- Words $\longleftrightarrow$ Definitions
- Sentences $\longleftrightarrow$ Theorems

(Roughly)

Need words to express ideas
Words have no intrinsic reality;
their worth comes from their usefulness.

- Does “dog” exist? Separates “dog food” from “cat food”
- Does “$\sqrt{-1}$” exist?

Wrong Question!

Is $\sqrt{-1}$ Useful?

Real Question

What concepts does $\sqrt{-1}$ allow us to talk about?

- Roots of any polynomial, e.g. $x^2 + 1$
  Seems boring, but incredibly useful
- Signal analysis
- Electromagnetism
- Quantum Theory
- . . .

The shortest path between two truths in the real domain passes through the complex plane.

Hadamard

But How?

Complex Numbers

$=$

Real Numbers

$+$

$\sqrt{-1}$

Is that it?

Of course not!

Form Versus Function

Slogan

Mathematical objects are determined more by what they do than what they are.

Dynamic not static.

What can we do with numbers?

1. Add: $2 + 3 = 5$, $\pi + e = 5.8599\ldots$
2. Subtract: $3 - 2 = 1$, $\pi - e = 0.42331\ldots$
3. Multiply: $2 \times 3 = 6$, $\pi \times e = 8.5397\ldots$
4. Divide (if non-zero): $6 \div 3 = 2$, $\pi \div e = 1.1557\ldots$
5. Compare: $2 \leq 3$, $\pi \geq e$
6. Subject to (lots of) rules.

We'd like to do these with complex numbers.

Notational Pause

Warning

Notation $\sqrt{-1}$ is unsafe!

Problem

Already in $\mathbb{R}$, square roots are not unique.

Solution

Don't add $\sqrt{-1}$.

Add i and add the rule $\mathrm{i}^2 = -1$.

Set of Complex Numbers

Written as $\mathbb{C}$.

Generating Complex Numbers

- Start: $\mathbb{R} \subseteq \mathbb{C}$ and $\mathrm{i} \in \mathbb{C}$.
- Add: $3 + \mathrm{i} \in \mathbb{C}$, $x + \mathrm{i} \in \mathbb{C}$,
- Multiply: $\mathrm{i}2 \in \mathbb{C}$, $\mathrm{i}y \in \mathbb{C}$,
- Back to Add: $3 + \mathrm{i}2 \in \mathbb{C}$, $x + \mathrm{i}y \in \mathbb{C}$.

Story So Far

$$x, y \in \mathbb{R} \Longrightarrow x + \mathrm{i}y \in \mathbb{C}$$

Examples

1. $2 + \mathrm{i}3$
2. $1 + \mathrm{i}(-1)$
3. $\pi + \mathrm{i}e$

What Else?

Question

What else must we have?
(So that $+$, $-$, $\times$, $\div$ defined)

Answer

Nothing!
Because of the rules.

Rules? What rules?

Same as $\mathbb{R}$.

Addition

$$\begin{aligned}(\mathrm{e}+\mathrm{i}3)+(\pi+\mathrm{i}2) &= (\mathrm{e}+\mathrm{i}3)+(\mathrm{i}2+\pi)\\ &= \mathrm{e}+(\mathrm{i}3+(\mathrm{i}2+\pi))\\ &= \mathrm{e}+((\mathrm{i}3+\mathrm{i}2)+\pi)\\ &= \mathrm{e}+(\pi+\mathrm{i}(3+2))\\ &= (\mathrm{e}+\pi)+\mathrm{i}5\end{aligned}$$

Of the form $x+\mathrm{i}y$ with $x, y \in \mathbb{R}$.

General Rule

$$(x+\mathrm{i}y)+(u+\mathrm{i}v) = (x+u)+\mathrm{i}(y+v)$$

Subtraction

$$\begin{aligned}(\mathrm{e}+\mathrm{i}3)-(\pi+\mathrm{i}2) &= (\mathrm{e}+\mathrm{i}3)+((-\pi)+\mathrm{i}(-2)) \\ &= (\mathrm{e}-\pi)+\mathrm{i}(3-2) \\ &= (\mathrm{e}-\pi)+\mathrm{i}1 \\ &= (\mathrm{e}-\pi)+\mathrm{i}\end{aligned}$$

Of the form $x+\mathrm{i}y$ with $x, y \in \mathbb{R}$.

General Rule

$$(x+\mathrm{i}y)-(u+\mathrm{i}v) = (x-u)+\mathrm{i}(y-v)$$

Multiplication

$$\begin{aligned}(\mathrm{e}+\mathrm{i}3)(\pi+\mathrm{i}2) &= \mathrm{e}\pi+\mathrm{e}\mathrm{i}2+\mathrm{i}3\pi+\mathrm{i}3\mathrm{i}2\\ &= \mathrm{e}\pi+\mathrm{i}\mathrm{e}2+\mathrm{i}3\pi+\mathrm{i}^2 6\\ &= \mathrm{e}\pi+\mathrm{i}(\mathrm{e}2+3\pi)+\mathrm{i}^2 6\\ &= \mathrm{e}\pi+\mathrm{i}(\mathrm{e}2+3\pi)+-1\times 6\\ &= (\mathrm{e}\pi-6)+\mathrm{i}(\mathrm{e}2+3\pi)\end{aligned}$$

General Rule

$$(x+\mathrm{i}y)(u+\mathrm{i}v) = (xu-yv)+\mathrm{i}(xv+yu)$$

Division

Question

What is division?

Answer

$$\frac{1}{x} \times x = 1$$

Question

What is 1?

Answer

$$1 \times x = x$$

Division

- Finding 1: solve

$$(x + \mathrm{i}y)(u + \mathrm{i}v) = (u + \mathrm{i}v)$$

- Specific instance: solve

$$(x + \mathrm{i}y)(2 + \mathrm{i}3) = 2 + \mathrm{i}3$$

- Expand out multiplication:

$$2x - 3y + \mathrm{i}(2y + 3x) = 2 + \mathrm{i}3$$

- Key Fact: $u + \mathrm{i}v = s + \mathrm{i}t \iff u = s$ and $v = t$
- Solve: $2x - 3y = 2$ and $2y + 3x = 3$
- Solution: $x = 1$, $y = 0$.
- Conclusion: $1 = 1 + \mathrm{i}0$ (unsurprisingly)

Division

- Finding $1/(x + \mathrm{i}y)$: solve

$$(x + \mathrm{i}y)(u + \mathrm{i}v) = 1 + \mathrm{i}0$$

- Specific instance: solve

$$(2 + \mathrm{i}3)(u + \mathrm{i}v) = 1 + \mathrm{i}0$$

- Expand out multiplication:

$$2u - 3v + \mathrm{i}(2v + 3u) = 1 + \mathrm{i}0$$

- Solve: $2u - 3v = 1$ and $2v + 3u = 0$
- Substitute: $v = -3u/2$
- Get: $u(2 + 9/2) = 1$
- Solution: $u = 2/13$, $v = -3/13$.

Division

General Formula

$x + \mathrm{i}y \neq 0$ then

$$\frac{1}{x + \mathrm{i}y} = \frac{x}{x^2 + y^2} - \mathrm{i}\frac{y}{x^2 + y^2}$$

Looks a bit scary . . . split it up:

$$\frac{1}{x + \mathrm{i}y} = \frac{1}{x^2 + y^2}(x - \mathrm{i}y)$$

Story So Far

Complex Numbers

- Notation $\mathbb{C}$
- $z \in \mathbb{C}$ represented as $z = x + \mathrm{i}y$ with $x, y \in \mathbb{R}$
- Add, Subtract, Multiply, Divide
- Same rules as in $\mathbb{R}$ with . . .
- $\mathrm{i}^2 = -1$
- $\mathbb{R} \subseteq \mathbb{C}$ as $x \mapsto x + \mathrm{i}0$
- Conventions: leave out 1 or 0 if unambiguous:

$$2 + \mathrm{i}0 \to 2, \qquad 2 + \mathrm{i}1 \to 2 + \mathrm{i}$$

Complex Conjugation

Division formula:

$$\frac{1}{x + \mathrm{i}y} = \frac{1}{x^2 + y^2}(x - \mathrm{i}y)$$

$x - \mathrm{i}y$ is the complex conjugate of $x + \mathrm{i}y$
written: $\overline{x + \mathrm{i}y}$

Examples

- $\overline{2 + \mathrm{i}3} = 2 - \mathrm{i}3$
- $\overline{3} = \overline{3 + \mathrm{i}0} = 3 - \mathrm{i}0 = 3$
- $\overline{\mathrm{i}4} = -\mathrm{i}4$

Real and Imaginary Parts

$$z = x + \mathrm{i}y$$

- x is the real part of z

$$x = \operatorname{Re} z = \frac{1}{2}(z + \overline{z})$$

$$\operatorname{Re}(2 + \mathrm{i}3) = 2$$

- y is the imaginary part of z

$$y = \operatorname{Im} z = \frac{1}{2}(z - \overline{z})$$

$$\operatorname{Im}(2 + \mathrm{i}3) = 3$$

Summary

- Complex numbers are like real numbers with $\mathbf{i}$
- Add, Subtract, Multiply, Divide just as in $\mathbb{R}$
- New thing: complex conjugation

Things To Ponder

- Can we compare complex numbers? $z \leq w$?
- Gaussian integers: all complex numbers of form $n + \mathbf{i}m$. How does this compare to $\mathbb{Z}$?