How to solve a Linear Algebra problem.

1/ Make a Matrix

2/ Do Gauss Elimination

3/ Interpret the answer

1 Make a Matrix

(dzc09, 4/ G. E.)

- Is there a matrix already there?
- Do we use that one, or another?

$$\underline{\underline{Ax = b}}$$

Run Gauss on $[A \mid b]$

$$\underline{\underline{Ax = 0}}$$

Run Gauss on A

Basis of Null(A)

Gauss on A

Basis of Col(A)

Gauss on A

$$\mathrm{Null}(A)^{\perp} = \mathrm{Col}(A^T) = \mathrm{Row}(A)$$

$$\mathrm{Null}(A^T) = \mathrm{Col}(A)^{\perp}$$

(Orthogonal projection onto Col(A)

Method GS

- Run GS on A to get orthogonal basis
- Compute $\sum_i \frac{\langle v_i, b\rangle v_i}{\langle v_i, v_i\rangle}$

Method GE

Project b onto Col(A)

$$\boxed{A^T A x = A^T b}$$

$$\left(\left(A^T A x = A^T b\right)\right)$$

Solve using Gauss

Write down Ax.

(dec 09, 4)

$$\begin{bmatrix} 1 & -3 & 0 & 1 & 0 \\ -2 & 6 & -2 & 0 & -3 \\ 1 & -3 & 6 & -5 & 9 \end{bmatrix}$$

$$\xrightarrow{\substack{R_2 + 2R_1 \\ R_3 - R_1}} \begin{bmatrix} 1 & -3 & 0 & 1 & 0 \\ 0 & 0 & -2 & 2 & -3 \\ 0 & 0 & 6 & -6 & 9 \end{bmatrix}$$

$$\xrightarrow{R_3 + 3R_2} \begin{bmatrix} 1 & -3 & 0 & 1 & 0 \\ 0 & 0 & -2 & 2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 \\ -3 & 6 & -3 \\ 0 & -2 & 6 \\ 1 & 0 & -5 \\ 0 & -3 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & -2 & 6 \\ 0 & 2 & -6 \\ 0 & -3 & 9 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -2 & 1 \\ 0 & -2 & 6 \\ 0 & 0 & 0 \\ 0 & 2 & -6 \\ 0 & -3 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 1 \\ 0 & -2 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

<u>Null(A)</u>

$Ax = 0$

$\text{ref}(A)x = 0$

$$\begin{bmatrix} 3r - s \\ r \\ s - \frac{3}{2}t \\ s \\ t \end{bmatrix}$$

<u>basis:</u> $\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -3/2 \\ 0 \\ 1 \end{bmatrix}$

Col(A)

Columns in A corresponding to pivot cols in rref(A)

$$\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ -2 \\ 6 \end{bmatrix}$$

$\underline{\text{Row}(A)} = \text{Col}(A^T)$

~~to~~ $\text{ref}(A^T)$

$$\begin{bmatrix} 1 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 6 \\ -2 \\ 0 \\ -3 \end{bmatrix}$$

(can be read off from $\text{ref}(A^T)$)

$$\text{Col}(A)^{\perp} = \text{Null}(A^T)$$

$$\begin{bmatrix} 5s \\ 3s \\ s \end{bmatrix} \quad \text{basis} \quad \begin{bmatrix} 6 \\ 3 \\ 1 \end{bmatrix}$$

Orthogonal Projection

$$A^T A x = A^T b$$

$$\begin{bmatrix} 1 & -2 & 1 \\ -3 & 6 & -3 \\ 0 & -2 & 6 \\ 1 & 0 & -5 \\ 0 & -3 & 9 \end{bmatrix} \begin{bmatrix} 1 & -3 & 0 & 1 & 0 \\ -2 & 6 & -2 & 0 & -3 \\ 1 & -3 & 6 & -5 & 9 \end{bmatrix} = \begin{bmatrix} \quad & & \\ & & \\ & & \end{bmatrix}$$

Solve for x,
apply A to get
closest point.

Eigenvalue Problems (Dec 09, 6)

$$\begin{bmatrix} 7 & 24 \\ 24 & -7 \end{bmatrix}$$

$$t^2 - \operatorname{tr}A\, t + \det A$$

$$t^2 - 625$$

$$t = \pm 25$$

$$(A - tI)x = 0$$

25

$$\begin{bmatrix} -18 & 24 \\ 24 & -32 \end{bmatrix} x = 0 \qquad \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 4 \\ 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 4/\sqrt{5} \\ 3/\sqrt{5} \end{bmatrix} \qquad 2\sqrt{5}$$

$$\begin{bmatrix} -3 \\ 4 \end{bmatrix} \longrightarrow \begin{bmatrix} -3/\sqrt{5} \\ 4/\sqrt{5} \end{bmatrix} \qquad -2\sqrt{5}$$

Theoretical Problems

- Where do I start?
    - A square
    - A invertible
- Where am I going?
    - $A^T A$ invertible

- Expand unfamiliar concepts
  - Find equivalent statements

In this case long list for A & A^TA

A is invertible if A is square and:

1/ there is a B such that $AB = I = BA$

2/ $Ax = 0$ has only 1 solution

3/ $Ax = b$ has a solution for every b

4/ $\mathrm{rref}(A) = I$

5/ $\det(A) \neq 0$

6/ A has full rank

Other facts:

C, D invertible $\Rightarrow$ CD invertible

Q: involves A and A^TA.

Write out lists for A & A^TA

A	A^TA
• $\exists B$ st $AB = I = BA$	$\exists C$ st $A^TAC = I = CA^TA$
• $Ax = 0$ unique	$A^TAx = 0$ unique
• $Ax = b$ exists	$A^TAx = b$ exists
• rref $A = I$	rref $A^TA = I$
• $\det A \neq 0$	$\det A^TA \neq 0$
• A has full rank	A^TA has full rank

Now look for connections

Lots of connections exist, simplest is probably with determinants:

$$\det A \neq 0 \iff \det A^TA \neq 0$$

- have $\det A^TA = \det A^T \det A$

  and $\det A^T = \det A$

  so $\det A^TA = (\det A)^2$

Hence $\det A^TA \neq 0$ iff $\det A \neq 0$

so A^TA invertible iff A invertible