TMA4115 Matematikk 3

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Lecture 15: Euclidean Space, Sweet Euclidean Space

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Key Points

- "Vector spaces" are the right "homes" for the inputs and outputs of matrices
- "Vector spaces" are spaces that "look like $\mathbb{R}^n$"

Recap

- Special Processes $\longleftrightarrow$ Matrices

Static versus Dynamic

Matrices	Special Processes
$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 4 & 6 \end{bmatrix}$	$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} x_1 + 2x_2 \\ 3x_1 + 4x_2 \\ 4x_1 + 6x_2 \end{bmatrix}$
Data	Function
Representation	Actuality
Static	Dynamic

Implementation

```
m-vector apply_matrix_A (n-vector x)
{
  m-vector y;
  for (i=1; i<=m; i++) {
     y.i = 0;
     for (j=1; j<=n; j++) {
       y.i += A.i.j*x.j;
       }
     }
  }
  return y;
}
```


Implementation of “Do Nothing”

```
do_nothing(x)
{
  return x;
}
```

With extremely strict typing:

```
type do_nothing(type x)
{
  return x;
}
```

- For each `type`, a **different** process.
- For each n, a **different** identity matrix.

Implementation of “Say Sausages”

```
say_sausages(x)
{
  return ‘‘sausages’’;
}
```

With extremely strict typing:

```
food say_sausages(type x)
{
  return ‘‘sausages’’;
}
```

- For each `type`, a **different** process.

Mathematical Data Typing

Rough Definition

A
set
or
mathematical data type
is a collection of elements that can be precisely defined

i.e. the question:
"is x in X?" ($x \in X$?)
always returns TRUE or FALSE

A process always has a set of
valid inputs
and
possible outputs

Typing Wrapper

```
boolean can_I_do_function (function f, element x)
{
  if (f.valid_inputs = x.containing_set)
  {
     return TRUE;
  }
  else
  {
     return FALSE;
  }
}
```

Caution
Always **declare** your variables!

A Home for Vectors

Question

1. What is the right set of "valid inputs" for a matrix?
2. What is the right set of "possible outputs" for a matrix?

Answer

1. All vectors **of the right length**
2. All vectors **of the right length**

Note: **possible** outputs not **actual** outputs.

Vectors

Question

What is a vector?

Answer

An ordered list of n real numbers.

Example

$$(1,2,3,4),\ (\pi, e, \pi^e, e^\pi)$$

Exact notation not important so long as **order** is clear.

$$(1,2),\quad \begin{bmatrix}1\\2\end{bmatrix},\quad \begin{matrix}2\\1\end{matrix}$$

Euclidean Space

Definition

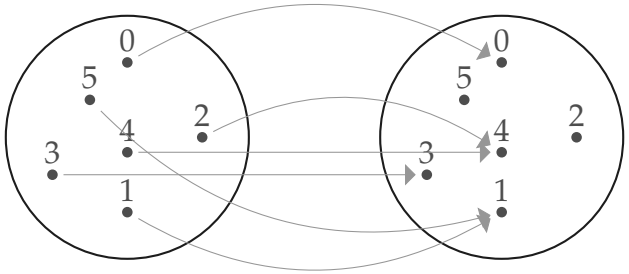
***n*–dimensional (Euclidean) space**
is the set of all vectors (of real numbers) of length n

Notation: $\mathbb{R}^n$

- EP: n–dimensional space (bit vague)
- Euclidean **can** imply distance as well
- "$\mathbf{x} \in \mathbb{R}^n$" means "let $\mathbf{x}$ be an ordered list of n real numbers"
- "$\mathbf{x} \in \mathbb{R}^n$?" means "is $\mathbf{x}$ an ordered list of n real numbers?"

. . . The Company That He Keeps

Idea
Use vectors to study matrices.



$n \mapsto n^2 \mod 6$

Structure Reveals Structure

Idea
Use structure of $\mathbb{R}^n$ to study matrices.

What structure is on $\mathbb{R}$?

1. Addition: $x + y$
2. Subtraction: $x - y$
3. Multiplication: $x \cdot y$
4. Additive Inverse: $-x$
5. Multiplicative Inverse: x^{-1}
6. Special Elements: 0, 1
7. Length: $|x|$
8. Scaling: λx
9. Comparision: $x \geq y$
10. More?

Structure Reveals Structure

Useful Structure
Useful structure: interacts nicely with matrices.

A 1×1–matrix is just . . . a number! $\begin{bmatrix} a \end{bmatrix}$ is $x \mapsto ax$

1. **Addition:** $x + y$ — $ax + ay = a(x + y)$
2. **Subtraction:** $x - y$ — $ax - ay = a(x - y)$
3. Multiplication: $x \cdot y$ — $ax \cdot ay = a^2(x \cdot y)$
4. **Additive Inverse:** $-x$ — $-ax = a(-x)$
5. Multiplicative Inverse: x^{-1} — $(ax)^{-1} = a^{-1}x^{-1}$
6. **Special Elements:** 0, (1) — $a0 = 0$, $a1 = a$
7. Length: $|x|$ — $|ax| = |a||x|$
8. **Scaling:** λx — $\lambda(ax) = a(\lambda x)$
9. Comparision: $x \geq y$ — $ax \geq ay$ or $ax \leq ay$)

More Generally

Define addition, scaling, zero **componentwise** on vectors:

$$\begin{bmatrix}1\\2\\3\end{bmatrix}+\begin{bmatrix}4\\5\\6\end{bmatrix}=\begin{bmatrix}5\\7\\9\end{bmatrix}$$

$$3\begin{bmatrix}1\\2\\3\end{bmatrix}=\begin{bmatrix}3\\6\\9\end{bmatrix}$$

$$\mathbf{0}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Note: fix size, then do operations.

Useful Structure

Addition, Subtraction, $\mathbf{0}$, Scaling
commute
with applying matrices

For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$

$$A(\mathbf{x}+\mathbf{y}) = A\mathbf{x} + A\mathbf{y}$$
$$A(\lambda\mathbf{x}) = \lambda A\mathbf{x}$$
$$A\mathbf{0} = \mathbf{0}$$

Useful Structure

Operations

- Addition: $\mathbf{x}+\mathbf{y}$
- Scaling: $\lambda\mathbf{x}$
- Zero Vector: $\mathbf{0}$

Identities

Associative	$\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$
Commutative	$\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$
Identity	$\mathbf{v}+\mathbf{0}=\mathbf{v}$
Inverses	$\mathbf{v}+(-\mathbf{v})=\mathbf{0}$
Distributivity	$\lambda(\mathbf{u}+\mathbf{v})=\lambda\mathbf{u}+\lambda\mathbf{v}$
Compatibility (multiplication)	$\lambda(\mu\mathbf{v})=(\lambda\mu)\mathbf{v}$
Compatibility (addition)	$(\lambda+\mu)\mathbf{v}=\lambda\mathbf{v}+\mu\mathbf{v}$
Compatibility (unit)	$1\mathbf{v}=\mathbf{v}$

Aide–Mémoire

Addition, Scaling, Zero behave just like in $\mathbb{R}$.

Some Special Vectors

Introducing...

$$\mathbf{e}_j = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$\longleftarrow$ jth

The **elementary** vectors.

Special Property One

Lemma
Every vector is a unique sum of scales of elementary vectors.

Proof.

$$\begin{aligned} \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} &= x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + x_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \\ &= x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n \end{aligned}$$

□

Examples

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \pi \\ \mathrm{e} \\ 0 \\ 2 \end{bmatrix} = \pi\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \mathrm{e}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Special Property Two

Lemma
A matrix is completely determined by what it does to the elementary vectors.

Proof.

$$\begin{aligned} A\mathbf{x} &= A(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n) \\ &= x_1A\mathbf{e}_1 + x_2A\mathbf{e}_2 + \cdots + x_nA\mathbf{e}_n \end{aligned}$$

□

Example

$$\begin{aligned}\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} &= \begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\left(x\begin{bmatrix}1\\ 0\end{bmatrix} + y\begin{bmatrix}0\\ 1\end{bmatrix}\right)\\ &= x\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} + y\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix}\\ &= x\begin{bmatrix}1\\ 3\end{bmatrix} + y\begin{bmatrix}2\\ 4\end{bmatrix}\end{aligned}$$

Summary

- Vectors add and scale like numbers
- Decompose arbitrary vectors using elementary vectors
- Understand matrices by applying to elementary vectors