TMA4115 Matematikk 3

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Lecture 13: MatrixirtaM

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Key Points

- Invertible matrices are useful
- Recognising invertible matrices

Recap

- Special Processes $\longleftrightarrow$ Matrices
- Invertible process:

  Input $\longleftrightarrow$ Output

Invertible Processes

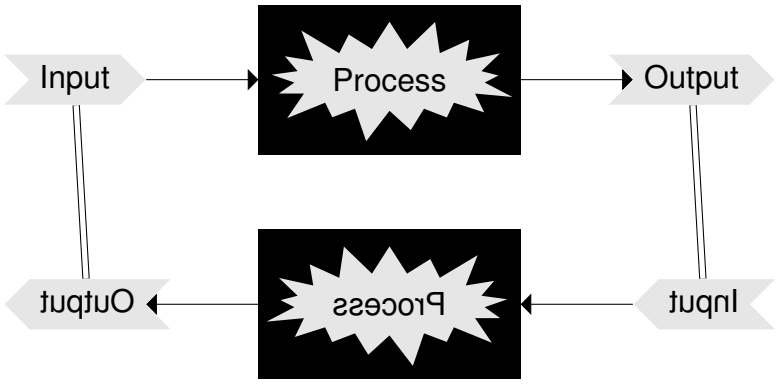
Invertible
A **process** is invertible if:

1. Each input is uniquely determined by its output
2. Each potential output is possible

Remark

1. A process is invertible $\Longleftrightarrow$ it can be **reversed**
2. There is another process which **reconstructs** the input from the output

Invertible Processes



Going all the way around gets you right back where you started.

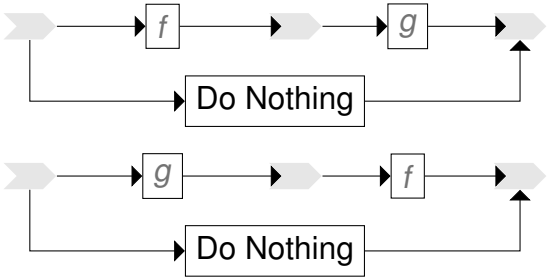
A Very Special Process

Introducing . . .

The
Do Nothing
Process!

The “Do Nothing” process . . . does nothing!

Busy Doing Nothing



f invertible
$\Longleftrightarrow$
$\exists g$: gf and fg are both
“Do Nothing”

In Matrix–Speak

The "Do Nothing" process:

$$y_1 = x_1, y_2 = x_2, \ldots, y_n = x_n$$

Representing matrix: (Technically, one for each size n)

$$I_n := \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Definition or Theorem

A matrix A is
invertible
if there is a matrix B such that
$AB = I_m, \qquad BA = I_n$

Why Bother?

Basic Problem

Solve

$$A\mathbf{x} = \mathbf{b}$$

Output of process A is $\mathbf{b}$, what was the input?

Obvious Use of Invertibility

$$BA\mathbf{x} = B\mathbf{b}$$

1. **If** A is invertible, inverse B ... apply B
2. Use $BA = I_n$
3. I_n means "do nothing"

The True Meaning of Invertible

Exercise

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Tasks:

1. Factor 1st number
2. Factor 2nd number
3. Multiply the numbers

Point

It is easier to **do** than **undo**

True Use of Invertibility

A invertible
$A\mathbf{b} = \mathbf{c}$
$\Longrightarrow$
$\mathbf{b}$ and $\mathbf{c}$ represent **same** data

Another View

$\mathbf{b}$ and $\mathbf{c}$ are
two ways
of seeing the same thing

Consequence

Use invertible matrices to change your point of view to something nicer

True Use of Invertibility

$$A\mathbf{x} = \mathbf{b}$$

Do Not
Try to find inverse of A — factoring

Do
Use invertible matrices so A looks nice — multiplying

Remember
Do the same to **both** sides!

Finding Invertible Matrices

Invertible

$$A \text{ invertible} \iff \exists B : AB = I_m \text{ and } BA = I_n$$

Difficulty
Proving invertible means finding B!

Obvious Examples

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}, \qquad I_n I_n = I_n$$

("do nothing" and "do nothing again" = "do nothing")

$$\begin{bmatrix} 0 & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}, \text{ has inverse } \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$(x_1, x_2, \ldots, x_n) \mapsto (x_2, x_1, \ldots, x_n)$$
reversed by
$$(y_1, y_2, \ldots, y_n) \mapsto (y_2, y_1, \ldots, y_n)$$

Examples

$$\begin{bmatrix} 2 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}, \text{ has inverse } \begin{bmatrix} 1/2 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$(x_1, x_2, \ldots, x_n) \mapsto (2x_1, x_2, \ldots, x_n)$$
reversed by
$$(y_1, y_2, \ldots, y_n) \mapsto (1/2 y_1, y_2, \ldots, y_n)$$

More generally, for $\lambda_i \neq 0$

$$\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}, \text{ has inverse } \begin{bmatrix} 1/\lambda_1 & 0 & \cdots & 0 \\ 0 & 1/\lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1/\lambda_n \end{bmatrix}$$

More Examples

$$\begin{bmatrix} 1 & 1 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}, \text{ has inverse } \begin{bmatrix} 1 & -1 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$(x_1, x_2, \ldots, x_n) \mapsto (x_1 + x_2, x_2, \ldots, x_n)$$
reversed by
$$(y_1, y_2, \ldots, y_n) \mapsto (y_1 - y_2, y_2, \ldots, y_n)$$

Row Operations are Invertible Matrices

Key Fact

Row op: $A \to B$
$\Longrightarrow$
$B = EA$
with
E invertible

Two Key Facts

1. There is only one way to invert an invertible process
   $\Longrightarrow$ if A is invertible, its inverse is unique
   Notation: A^{-1}
2. The composition of two invertible processes is invertible
   $\Longrightarrow$ if A, B invertible and composable, then AB is invertible
   $(AB)^{-1} = B^{-1}A^{-1}$
   ("take off shoes, **then** socks")

Important Implications

Row Reduction

If $A \to B$ via row reduction then
A invertible $\Longleftrightarrow$ B invertible

If $A \to I_m$ via row reduction then A is invertible

Can we say more?

The Result of Row Reduction

Full Gauss–Jordan Row Reduction

Example outcomes:

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & 8 \end{bmatrix}$$

Not invertible:

$$A \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & 8 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Inevitable **if** rows(A) < cols(A)

The Result of Row Reduction

Full Gauss–Jordan Row Reduction

Example outcomes:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Not invertible:

$$Ax = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

has no solution.

Inevitable **if** rows(A) > cols(A)

Important Implications

Invertible Size

$$A \text{ invertible} \implies A \text{ square}$$

Other direction false:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

is not invertible.

Square Row Reduction Results

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & * \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The first is the only matrix such that:

1. A is invertible
2. $A\mathbf{x} = \mathbf{b}$ has a solution for **all** $\mathbf{b}$
3. $A\mathbf{x} = \mathbf{b}$ has a unique solution for **some** $\mathbf{b}$
4. $A\mathbf{x} = \mathbf{0}$ has a unique solution

This is independent of size of square matrix

Other Methods

When is

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

invertible?
Can solve $A\mathbf{x} = \mathbf{b}$ for all $\mathbf{b}$
Apply GE to

$$\left[\begin{array}{cc|c} a & b & b_1 \\ c & d & b_2 \end{array}\right] \xrightarrow[a\neq 0]{R_2 - {}^{c}/_{a} R_1} \left[\begin{array}{cc|c} a & b & b_1 \\ 0 & d - {}^{cb}/_{a} & b_2 - {}^{c}/_{a} b_1 \end{array}\right]$$

$$\xrightarrow{R_2 \to aR_2} \left[\begin{array}{cc|c} a & b & b_1 \\ 0 & ad - cb & ab_2 - cb_1 \end{array}\right]$$

If $a \neq 0$ then A invertible $\iff ad - bc \neq 0$
If $c \neq 0$ then swap so A invertible $\iff cb - ad \neq 0$
If $a = c = 0$ then A **not** invertible **and** $cb - ad = 0$

It's that Strange Sense of Deja Vu

y_1, y_2 solutions of an ODE
linearly independent
$\iff$
$y_1 y_2' - y_1' y_2 \neq 0$

A
invertible
$\iff ad - bc \neq 0$

Not a coincidence!

Determining Invertibility

Definition

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

*the **determinant** of A is*

$$\det(A) := \begin{vmatrix} a & b \\ c & d \end{vmatrix} := ad - bc$$

Lemma

$$A \text{ invertible} \iff \det(A) \neq 0$$

Bigger and . . . Better?

Sort of Definition
Square matrices of any size have determinants.
Exact definition not exactly . . . memorable:

$$\det(A) = \sum_{\sigma \in \Sigma_n} (-1)^{\sigma} a_{11_\sigma} \cdots a_{nn_\sigma}$$

Best: Learn rules for computation.

Don't Know? Ask Gauss

Key Properties for Computation

1. $\det(I_n) = 1$
2. $\det(AB) = \det(A)\det(B)$
3. Triangular matrices:

$$\begin{vmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 7 \end{vmatrix} = 2 \times 5 \times 7 = 70$$

4. Row Swaps:

$$\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = - \begin{vmatrix} 4 & 5 \\ 2 & 3 \end{vmatrix}$$

5. Row Multipliers:

$$\begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix} = 2 \begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix}$$

Note: only **one** row affected.

Don't Know? Ask Gauss

6. Row Adds:

$$\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 0 & -1 \end{vmatrix}$$

No change!

7. Works for columns too.

Conclusion
Whenever you need to **compute** something, use GE!

Mnemonic

$$\begin{vmatrix} 0 & 1 \\ 2 & 2 \end{vmatrix} = -2$$

Example

$$\begin{vmatrix} 0 & 2 & 3 \\ 3 & 4 & 0 \\ 1 & 6 & 2 \end{vmatrix}$$

$$= - \begin{vmatrix} 1 & 6 & 2 \\ 0 & -14 & -6 \\ 0 & 2 & 3 \end{vmatrix} \quad (R_2 - 3R_1)$$

$$= 2 \begin{vmatrix} 1 & 3 & 2 \\ 0 & 1 & 3 \\ 0 & -7 & -6 \end{vmatrix} \quad (R_2 \leftrightarrow R_3)$$

$$= 2 \times 15 = 30$$

$$= - \begin{vmatrix} 1 & 6 & 2 \\ 3 & 4 & 0 \\ 0 & 2 & 3 \end{vmatrix} \quad (R_1 \leftrightarrow R_3)$$

$$= -2 \begin{vmatrix} 1 & 3 & 2 \\ 0 & -7 & -6 \\ 0 & 1 & 3 \end{vmatrix} \quad (C_2 \to \tfrac{1}{2} C_2)$$

$$= 2 \begin{vmatrix} 1 & 3 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 15 \end{vmatrix} \quad (R_3 + 7R_2)$$

Summary

- Use invertible matrices to **simplify** problems
- Invertible means "equivalent point of view"
- Lots of tests, most useful:
  A invertible $\iff$ A row reduces to I_n
- Computing determinants: use row and column rules to simplify