TMA4115 Matematikk 3

Andrew Stacey

Norges Teknisk-Naturvitenskapelige Universitet
Trondheim

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Lecture 12: Mathematics' Marvellous Matrices

Andrew Stacey

Norges Teknisk-Naturvitenskapelige Universitet
Trondheim

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Key Points

- Process $\longleftrightarrow$ Matrix:

  Obvious manipulations of processes

  $\longleftrightarrow$

  Matrix manipulations
- The true meaning of "invertible"

Detailed Recap

Definition

A
matrix
is a rectangular grid
(of numbers)

Example

$$\begin{bmatrix} 1 & 3 & & 5 \\ \pi & e^{\pi} & -\pi^{e} & -e \\ 0 & 28 & -2 & \end{bmatrix} \quad \checkmark$$

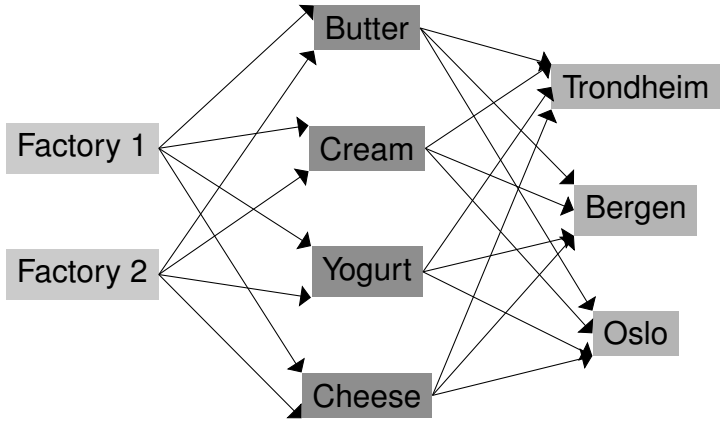
Meaningful Matrices

Key Point

Matrices **encode** certain processes

$$x_1, \ldots, x_n \longrightarrow \begin{matrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \qquad \ddots \qquad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{matrix}$$

Dairy Production



The Processes

Factories to Products

Products: how much of factory 2's output is butter?

$$\begin{bmatrix} & \text{factory 1} & \text{factory 2} \\ \text{butter} & .3 & .1 \\ \text{cream} & .1 & .0 \\ \text{yogurt} & .2 & .4 \\ \text{cheese} & .4 & .5 \end{bmatrix}$$

Products to Cities

How much of each tonne of butter goes to Trondheim?

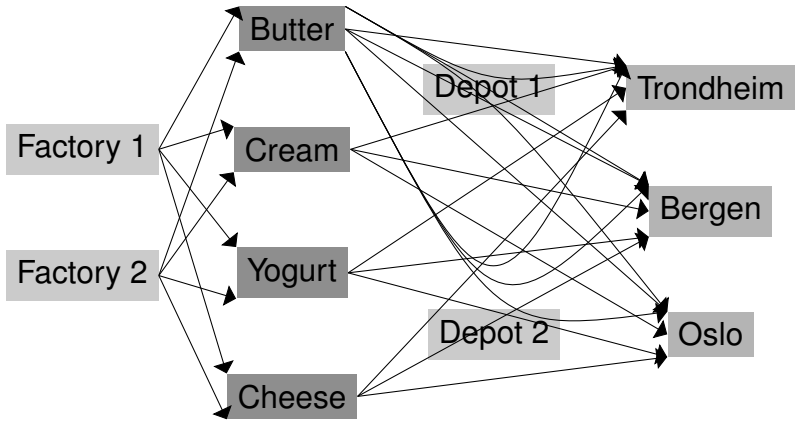
$$\begin{bmatrix} & \text{butter} & \text{cream} & \text{yogurt} & \text{cheese} \\ \text{Trondheim} & .22 & .25 & .27 & .21 \\ \text{Bergen} & .30 & .30 & .30 & .30 \\ \text{Oslo} & .48 & .45 & .43 & .49 \end{bmatrix}$$

From Factory to City

How much "stuff" goes from Factory 1 to Trondheim?

$$\begin{bmatrix} & \text{F 1} & \text{F 2} \\ \text{T} & .229 & .235 \\ \text{B} & .300 & .300 \\ \text{O} & .471 & .465 \end{bmatrix} = \begin{bmatrix} .22 & .25 & .27 & .21 \\ .30 & .30 & .30 & .30 \\ .48 & .45 & .43 & .49 \end{bmatrix} \begin{bmatrix} .3 & .1 \\ .1 & .0 \\ .2 & .4 \\ .4 & .5 \end{bmatrix}$$

Add in Depots



Partitioning Processes

Factory to City

via Depot 1

$$\begin{bmatrix} .087 & .105 \\ .087 & .105 \\ .116 & .140 \end{bmatrix}$$

via Depot 2

$$\begin{bmatrix} .142 & .030 \\ .213 & .195 \\ .355 & .325 \end{bmatrix}$$

Total

Factory 1 to Trondheim
=
Factory 1 to Trondheim via Depot 1
+
Factory 1 to Trondheim via Depot 2

$$\begin{bmatrix} .087 & .105 \\ .087 & .105 \\ .116 & .140 \end{bmatrix} + \begin{bmatrix} .142 & .030 \\ .213 & .195 \\ .355 & .325 \end{bmatrix} = \begin{bmatrix} .229 & .235 \\ .300 & .300 \\ .471 & .465 \end{bmatrix}$$

Combining Processes

1. One process follows the other: multiply matrices
2. Two processes **partition** another: add matrices

A Secret Process

Question
How do we increase factory output?

Answer
Increase Milk Production!

- Factory output proportional to milk production
- View factory output, (p_1, p_2), as a **process**:

$$\text{Milk} \longrightarrow \text{Produce}$$

- Representing matrix is:

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$$

Vectors

Definition

*A **column vector***
is a matrix with one column

Remark

1. Column vectors written as $\vec{u}$ or $\mathbf{u}$ or $\underline{u}$
2. View as "initial process": defining the input data for a process
3. Matrix $\times$ column vector = column vector
because output of one process is input of next

Matrices and Column Vectors

$$\begin{bmatrix} 1 \\ -10 \\ 10 \end{bmatrix} = \begin{bmatrix} 1 & 3 & -4 & 5 \\ 2 & 8 & 0 & 0 \\ 2 & 1 & -3 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}$$

Column Vectors and Linear Systems

Linear System

Find x, y, z such that
$$\begin{aligned} x + 2y + z &= 2 \\ 2x - 3y + z &= 3 \\ -x + y + 2z &= 4 \end{aligned}$$

Matrices

Find $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that $\begin{bmatrix} 1 & 2 & 1 \\ 2 & -3 & 1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$

Same question!

A Convenient Notation

Write generic linear system as:

$$A\vec{x} = \vec{b}$$

Composition order:

- A and B represent processes
- feed outputs of A into B to get new process, matrix C
- write $C = BA$
- substitute into notation for linear system:

$$C\vec{x} = BA\vec{x}$$

do A **first** and then B

Factory to Depot

Passing Through the Depots

$$\begin{bmatrix} & F1 & F2 \\ D1 & .29 & .35 \\ D2 & .71 & .65 \end{bmatrix}$$

Recall: matrix records **proportions**

Question
Can we find out the productivity of the factories by measuring what arrives at the depots?

Solve:

$$.29p_1 + .35p_2 = d_1$$
$$.71p_1 + .65p_2 = d_2$$

Solution

$$\left[\begin{array}{cc|c} .29 & .35 & d_1 \\ .71 & .65 & d_2 \end{array}\right] \xrightarrow{R_2 \to .29R_2} \left[\begin{array}{cc|c} .29 & .35 & d_1 \\ .2059 & .1885 & .29d_2 \end{array}\right]$$

$$\xrightarrow{R_2 - .71R_1} \left[\begin{array}{cc|c} .29 & .35 & d_1 \\ 0 & -.06 & .29d_2 - .71d_1 \end{array}\right]$$

$$\xrightarrow{R_1 \to -.06R_1} \left[\begin{array}{cc|c} -.0174 & -.0210 & -.06d_1 \\ 0 & -.06 & .29d_2 - .71d_1 \end{array}\right]$$

$$\xrightarrow{R_1 - .35R_2} \left[\begin{array}{cc|c} -.0174 & 0 & -.1885d_1 - .1015d_2 \\ 0 & -.06 & .29d_2 - .71d_1 \end{array}\right]$$

$$\xrightarrow{R_1 \to .29^{-1}R_1} \left[\begin{array}{cc|c} -.06 & 0 & .65d_1 - .35d_2 \\ 0 & -.06 & .29d_2 - .71d_1 \end{array}\right]$$

Back and Forth

$$.29p_1 + .35p_2 = d_1 \qquad \frac{-100}{6}\Big(.65d_1 - .35d_2\Big) = p_1$$
$$.71p_1 + .65p_2 = d_2 \qquad \frac{-100}{6}\Big(-.71d_1 + .29d_2\Big) = p_2$$

Remark

1. Can figure out what leaves factories from what arrives at depots.
2. Can arrange for any desired arrival amounts by adjusting productions.
3. No presumption of cause or effect.

Invertible

Definition

A process is said to be
invertible
if

1. *Each input is uniquely determined by its output*
2. *Each potential output is possible*

A matrix is
invertible
if
it represents an invertible process

Invertible Examples

1. $$\begin{bmatrix} .29 & .35 \\ .71 & .65 \end{bmatrix}$$
   represents an invertible process
2. $$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$
   is not invertible: $(0,0,0)$ and $(1,-2,1)$ both go to $(0,0)$
3. $$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
   is not invertible: no way to get $(1,0)$

An Incredibly Important Process

Introducing

The "do nothing" process!

Output = Input

$$y_1 = x_1, y_2 = x_2, \ldots, y_n = x_n$$

Representing matrix:

$$I_n := \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

(Technically, one for each size n)

Inverses

Rough Definition

An **inverse** of a process is another process such that composing **either way around** results in the "do nothing" process.

In matrix language: an inverse of A is B such that $AB = I_m$ and $BA = I_n$.

Lemma

A matrix is invertible if and only if it has an inverse.

Computing Inverses

Does

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 6 \end{bmatrix}$$

have an inverse?

If so, $A\vec{x} = \vec{b}$ has a solution for **any** $\vec{b}$.

For example ... $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Let's see if it does.

Simultaneous Gaussian Elimination

$$\begin{bmatrix}1&2&3&1&0&0\\2&3&4&0&1&0\\3&4&6&0&0&1\end{bmatrix} \begin{matrix}R_2-2R_1\\ \longrightarrow\\ R_3-3R_1\end{matrix} \begin{bmatrix}1&2&3&1&0&0\\0&-2&-2&-2&1&0\\0&-2&-3&-3&0&1\end{bmatrix}$$

$$\begin{matrix}R_3-2R_2\\ \longrightarrow\\ R_1+2R_2\end{matrix} \begin{bmatrix}1&0&-1&-3&2&0\\0&-1&-2&-2&1&0\\0&0&1&1&-2&1\end{bmatrix}$$

$$\begin{matrix}R_2+2R_3\\ \longrightarrow\\ R_1+R_3\end{matrix} \begin{bmatrix}1&0&0&-2&0&1\\0&-1&0&0&-3&2\\0&0&1&1&-2&1\end{bmatrix}$$

$$\begin{matrix}R_2\to -R_2\\ \longrightarrow\end{matrix} \begin{bmatrix}1&0&0&-2&0&1\\0&1&0&0&3&-2\\0&0&1&1&-2&1\end{bmatrix}$$

Check Your Answer

Is

$$B = \begin{bmatrix}-2&0&1\\0&3&-2\\1&-2&1\end{bmatrix}$$

an inverse for A?
Check: $AB = I_3$ **and** $BA = I_3$

Summary

- Matrix manipulations follow from what happens to processes
- Invertible matrices correspond to invertible processes
- Invertible process means input and output determine each other