TMA4115 Matematikk 3

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Lecture 11: What Is . . . The Matrix?

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Recap

- Solve linear systems by **elimination**
- Make systematic to ensure well-defined

Key Points

- Loose Ends
- The Language of the Matrix

Loose End from ODEs

*The shortest path between two truths in the real domain passes **through** the complex plane.*

Hadamard

"through": Remember to Return to Reality!

$$y'' + y = 0$$

Solutions: e^{ix}, e^{-ix} so $c_1e^{ix} + c_2e^{-ix}$, $c_1, c_2 \in \mathbb{C}$

Claim: $\big(\text{ODE}, y(0), y'(0)\big) \in \mathbb{R} \Longrightarrow y(t) \in \mathbb{R}$ all t

$$\begin{aligned} c_1e^{ix} + c_2e^{-ix} &= (a_1 + ib_1)(\cos(x) + i\sin(x)) \\ &\quad + (a_2 + ib_2)(\cos(x) - i\sin(x)) \\ &= (a_1 + a_2)\cos(x) + (b_2 - b_1)\sin(x) + i\text{junk} \\ &= A\cos(x) + B\sin(x) + i\text{junk} \end{aligned}$$

Real Solutions: $y(x) = A\cos(x) + B\sin(x)$

Loose Ends: Gaussian Elimination

- The Most Important Points:
  1. At each stage, do the obvious
  2. It works (theoretically)
  3. A computer can do it better
- In more detail:
  1. Goal: use row operations to get **lots** of zeros
  2. Don't mess with zeros **you have already made**
  3. Work systematically to ensure termination
- Most common mistakes:
  1. Work on **whole** system: LHS **and** RHS
  2. Signs: e.g. $-x$ becomes -1 in matrix
  3. Computational errors

GE: An Algorithm (for a person)

1. Convert linear system to matrix form

$$\begin{bmatrix} 2x + & 3y + & z = & -2 \\ 2x - & 3y + & 4z = & 1 \\ -1x + & 2y - & z = & -2 \end{bmatrix}$$

2. Use row operations to reduce matrix
   - **2.1** Work from left to right
   - **2.2** Work from top to bottom
   - **2.3** Leave alone rows already used
3. Interpret answer correctly
   - **3.1** Use "back substitution"
   - **3.2** Use "dummy variables" if have choice

GE: An Algorithm (pt 2: RREF)

```
matrix rref (matrix B)
{
  int l = 1;
  for (j=0; j<=colsize(B); j++) {
     for (i=l; i<=rowsize(B); i++) {
       if (B.i.j != 0) {
         rowswap(B,i,l);
         for (k=l+1; k<=rowsize(B); k++) {
           rowadd(B,k,l,-B.k.j/B.l.j);
         }
         l++;
         next j;
       }
     }
  }
  return B;
}
```

An Example

$$\begin{bmatrix} 2x+ & 3y+ & z & = -2 \\ 2x- & -3y+ & 4z & = 1 \\ -x+ & 2y- & z & = -2 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} -1 & 2 & -1 & -2 \\ 2 & -3 & 4 & 1 \\ 2 & 3 & 1 & -2 \end{bmatrix}$$

$$\xrightarrow[R_3+2R_1]{R_2+2R_1} \begin{bmatrix} -1 & 2 & -1 & -2 \\ 0 & 1 & 2 & -3 \\ 0 & 7 & -1 & -6 \end{bmatrix}$$

$$\xrightarrow{R_3-7R_2} \begin{bmatrix} -1 & 2 & -1 & -2 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & -15 & 15 \end{bmatrix}$$

Remarks

- Lazy algorithm: only alters **lower** rows.
  Can also eliminate from higher rows
  And renormalise

$$\begin{bmatrix} -1 & 2 & -1 & -2 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & -15 & 15 \end{bmatrix} \xrightarrow{R_1-2R_2} \begin{bmatrix} -1 & 0 & -5 & 4 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & -15 & 15 \end{bmatrix}$$

$$\xrightarrow[R_1-{}^1\!/_3 R_3]{R_2+{}^2\!/_{15}R_3} \begin{bmatrix} -1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -15 & 15 \end{bmatrix}$$

$$\xrightarrow[R_1 \to -R_1]{R_3 \to -{}^1\!/_{15}R_3} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

- Full reduction: **Gauss–Jordan**, partial: **Gauss**

Loose Ends: Number of Solutions

Theorem

$$2 \Longrightarrow \infty$$

Proof.

$y_1, \dots, y_n$ and $z_1, \dots, z_n$ both solutions of

$$a_1 x_1 + \cdots + a_n x_n = b$$

$\Longrightarrow (ty_1 + sz_1), \dots, (ty_n + sz_n)$ with $t + s = 1$ also solution:

$$\begin{aligned} &a_1(ty_1 + sz_1) + \cdots + a_n(ty_n + sz_n) \\ &\quad = t\big(a_1 y_1 + \cdots + a_n y_n\big) + s\big(a_1 z_1 + \cdots a_n z_n\big) \\ &\quad = tb + sb = b \quad \square \end{aligned}$$

Why Matrices

- Matrices **encode** linear systems
- Matrix $\longleftrightarrow$ Linear System

The Obligatory Matrix Stunt

$$\begin{bmatrix} 2x + & 3y + & z & = -2 \\ 2x - & 3y + & 4z & = 1 \\ -1x + & 2y - & z & = -2 \end{bmatrix}$$

The Rules

- Keep only the numbers
- Include "hidden" 0s, 1s, and -1s:

$$\text{“ ”} = 0x, \qquad x = 1x, \qquad -x = -1x$$

- Keep watch for signs

Why Linear Systems

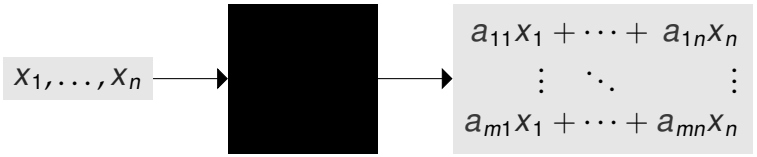
Mathematical problems are usually processes.

Question

1. I put in X, **what do I get?**
2. I got Y, **what did I start with?**
3. I put in X, I got Y, **what did I do?**

Why Linear Systems

Linear systems: processes of form:



Give this linear system the **label**:

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

Also note **size**: n inputs, m outputs: $m \times n$ matrix

Intermediate Step

What to do with processes

Compose them!

1. $x_1, \dots, x_n$ inputs to system
2. $z_1, \dots, z_m$ outputs
3. $y_1, \dots, y_k$ intermediate step

Assume:

1. $xs \to ys$ linear system
2. $ys \to zs$ linear system

Question

1. Is $xs \to zs$ a linear system?
2. If so, which?

Composition: Example

$$\begin{aligned} 3x_1 + x_2 &= y_1 \\ -2x_1 + 2x_2 &= y_2 \\ 4x_1 - x_2 &= y_3 \\ x_1 + x_2 &= y_4 \end{aligned} \qquad \begin{aligned} 3y_1 + 4y_2 + y_3 - y_4 &= z_1 \\ y_1 - 2y_2 + 2y_3 + 2y_4 &= z_2 \end{aligned}$$

Compose by substitution:

$$\begin{aligned} 3y_1 + 4y_2 + y_3 - y_4 &= z_1 \\ y_1 - 2y_2 + 2y_3 + 2y_4 &= z_2 \\ 3\big(3x_1 + x_2\big) + 4\big(-2x_1 + 2x_2\big) + \big(4x_1 - x_2\big) - \big(x_1 + x_2\big) &= z_1 \\ \big(3x_1 + x_2\big) - 2\big(-2x_1 + 2x_2\big) + 2\big(4x_1 - x_2\big) + 2\big(x_1 + x_2\big) &= z_2 \\ \big(9 - 8 + 4 - 1\big)x_1 + \big(3 + 8 - 1 - 1\big)x_2 &= z_1 \\ \big(3 + 4 + 8 + 2\big)x_1 + \big(1 - 4 - 2 + 2\big)x_2 &= z_2 \end{aligned}$$

Example Continued

$$\begin{aligned} 3x_1 + x_2 &= y_1 \\ -2x_1 + 2x_2 &= y_2 \\ 4x_1 - x_2 &= y_3 \\ x_1 + x_2 &= y_4 \end{aligned} \qquad \begin{aligned} 3y_1 + 4y_2 + y_3 - y_4 &= z_1 \\ y_1 - 2y_2 + 2y_3 + 2y_4 &= z_2 \end{aligned}$$

leads to

$$\begin{aligned} 4x_1 + 9x_2 &= z_1 \\ 17x_1 - 3x_2 &= z_2 \end{aligned}$$

- 4: amount of x_1 in z_1
  Computed via:

$$\begin{aligned} 3 \cdot 3 &: \left(\text{amount of } y_1 \text{ in } z_1\right) \text{ times } \left(\text{amount of } x_1 \text{ in } y_1\right) \\ +4 \cdot (-2) &: \left(\text{amount of } y_2 \text{ in } z_1\right) \text{ times } \left(\text{amount of } x_1 \text{ in } y_2\right) \\ +1 \cdot 4 &: \left(\text{amount of } y_3 \text{ in } z_1\right) \text{ times } \left(\text{amount of } x_1 \text{ in } y_3\right) \\ +(-1) \cdot 1 &: \left(\text{amount of } y_4 \text{ in } z_1\right) \text{ times } \left(\text{amount of } x_1 \text{ in } y_4\right) \end{aligned}$$

In Matrix–Speak

$$\left(x \xrightarrow{A} y\right)\begin{bmatrix} 3 & 1 \\ -2 & 2 \\ 4 & -1 \\ 1 & 1 \end{bmatrix} \qquad \left(y \xrightarrow{B} z\right)\begin{bmatrix} 3 & 4 & 1 & -1 \\ 1 & -2 & 2 & 2 \end{bmatrix}$$

The **composition** or **product** is

$$\left(x \xrightarrow{C} z\right)\begin{bmatrix} 4 & 9 \\ 17 & -3 \end{bmatrix}$$

Written in **this order**:

$$\begin{bmatrix} 4 & 9 \\ 17 & -3 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -2 & 2 \\ 4 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 4 & 1 & -1 \\ 1 & -2 & 2 & 2 \end{bmatrix}$$

Remarks

$$C = BA$$

- Order is by **convention**: learn it
  Idea: $f\big(g(x)\big)$ means **first** do g, **then** do f
  So **first** do A, **then** do B
- Must have: middle no. of variables same
  No. rows of A = no. cols of B
- Associative but **not** commutative
- Entries of C:

$$\begin{aligned} c_{ij} &= \text{amount of } x_j \text{ in } z_i \\ &= \sum_k \Big(\text{amount of } y_k \text{ in } z_i\Big) \times \Big(\text{amount of } x_j \text{ in } y_k\Big) \\ &= \sum_k b_{ik} a_{kj} \end{aligned}$$

In Pictures

$$\begin{bmatrix} 4 & 9 \\ 17 & -3 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 1 & -1 \\ 1 & -2 & 2 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -2 & 2 \\ 4 & -1 \\ 1 & 1 \end{bmatrix}$$

Summary

- Solve linear systems using elimination
- Linear systems model linear processes
- Composition of linear processes
  → Composition of linear systems
  → Multiplication of matrices