TMA4115 Matematikk 3

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Lecture 10: Elementary, my dear Gauss

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Recap

- Linear systems: things like

$$-x + 4y = 9$$
$$3x + 2y = 1$$

- Solve: means find numbers that make them **true**
- Some are easier to solve than others

Key Points

- Elementary operations change linear systems **without** changing solutions
- One other . . . to be revealed

Which Is Easier?

1. Solve
$$\begin{aligned} -x + 4y &= 9 \\ 3x + 2y &= 1 \end{aligned}$$
2. Solve
$$\begin{aligned} -x + 8 &= 9 \\ y &= 2 \end{aligned}$$

Key Idea
Solutions are **same** so transform **first** into **second**

Secondary Goal
Transform:
linear systems of 2nd type into linear systems of 1st type.

Simple Steps: Elementary Operations

1. Scaling: $\lambda \neq 0$
$$\begin{aligned} a_1x_1 + \ a_2x_2 + \ldots + \ a_nx_n &= \ b \\ \downarrow & \\ \lambda a_1x_1 + \lambda a_2x_2 + \ldots + \lambda a_nx_n &= \lambda b \end{aligned}$$
2. Swapping:
$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \downarrow & \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \end{aligned}$$

Simple Steps: Elementary Operations

3. Adding: any $\lambda \in \mathbb{R}$
$$\begin{aligned} a_{11}x_1 + \ & a_{12}x_2 + \ldots + \ & a_{1n}x_n = \ & b_1 \\ a_{21}x_1 + \ & a_{22}x_2 + \ldots + \ & a_{2n}x_n = \ & b_2 \\ & \downarrow & & \\ (a_{11}+\lambda a_{21})x_1 + & (a_{12}+\lambda a_{22})x_2 + \ldots + & (a_{1n}+\lambda a_{2n})x_n = & \ b_1+\lambda b_2 \\ a_{21}x_1 + \ & a_{22}x_2 + \ldots + \ & a_{2n}x_n = \ & b_2 \end{aligned}$$

Finding Solutions

What Makes It Work
In each case: if $x_1, x_2, \ldots, x_n$ satisfy 1st set, they **also** satisfy 2nd set.

1. Scaling: $\lambda \neq 0$
if x_1, x_2, …, x_n are such that
$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$
then x_1, x_2, …, x_n are such that
$$\begin{aligned} \lambda a_1x_1 + \lambda a_2x_2 + \cdots + \lambda a_nx_n & \\ = \lambda(a_1x_1 + a_2x_2 + \cdots + a_nx_n) & \\ = \lambda b & \end{aligned}$$

Finding Solutions

2. Swapping:
if $x_1, x_2, \ldots, x_n$ are such that

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \end{aligned}$$

then $x_1, x_2, \ldots, x_n$ are such that

$$\begin{aligned} a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \end{aligned}$$

Finding Solutions

3. Adding: any $\lambda \in \mathbb{R}$
if $x_1, x_2, \ldots, x_n$ are such that

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \end{aligned}$$

then $x_1, x_2, \ldots, x_n$ are such that

$$\begin{aligned} (a_{11}+\lambda a_{21})x_1 + (a_{12}+\lambda a_{22})x_2 + \ldots + (a_{1n}+\lambda a_{2n})x_n &= b_1+\lambda b_2 \\ a_{21}x_1 + \qquad a_{22}x_2 + \ldots + \qquad a_{2n}x_n &= \quad b_2 \end{aligned}$$

"If" versus "Iff"

Solution of 1st set **is** solution of 2nd.

Also want reverse: solution of 2nd **is** solution of 1st.

Comes from **reversibility**:

1. Scaling: scale by λ^{-1}
2. Swapping: swap back
3. Adding: add $-\lambda$ of 2nd to 1st

Examples

$$\begin{aligned} 2x + 3y + \ z &= -2 \\ 2x - 3y + 4z &= \ 1 \\ -x + 2y - \ z &= -2 \end{aligned} \xrightarrow{R_2-R_1} \begin{aligned} 2x + \ 3y + \ z &= -2 \\ -6y + 3z &= \ 3 \\ -x + \ 2y - \ z &= -2 \end{aligned}$$

$$\xrightarrow{R_3+\frac{1}{2}R_1} \begin{aligned} 2x + \ 3y + \ z &= -2 \\ -6y + \ 3z &= \ 3 \\ 3\tfrac{1}{2}y - \tfrac{1}{2}z &= -3 \end{aligned}$$

$$\xrightarrow{R_3+\frac{7}{12}R_2} \begin{aligned} 2x + \ 3y + \ z &= \ -2 \\ -6y + \ 3z &= \ 3 \\ 1\tfrac{1}{4}z &= -1\tfrac{1}{4} \end{aligned}$$

Stop here. Read off solution: $z = -1$, so $y = -1$, so $x = 1$

Slightly Simpler

$$\begin{array}{rl} 2x+3y+\ z=-2 & \\ 2x-3y+4z=\ 1 & \xrightarrow{R_1\leftrightarrow R_3} \\ -x+2y-\ z=-2 & \end{array} \qquad \begin{array}{r} -x+2y-\ z=-2 \\ 2x-3y+4z=\ 1 \\ 2x+3y+\ z=-2 \end{array}$$

$$\xrightarrow[R_3+2R_1]{R_2+2R_1} \qquad \begin{array}{r} -x+2y-\ z=-2 \\ y+2z=-3 \\ 7y-\ z=-6 \end{array}$$

$$\xrightarrow{R_3+7R_2} \qquad \begin{array}{r} -x+2y-\ z=-2 \\ y+\ 2z=-3 \\ -15z=15 \end{array}$$

Difference? No nasty fractions!

The Key Step

1. Scaling
2. Swapping
3. Adding

Reason
It's the only one that **substantially** alters the system.

Project

Design a "program" to solve linear systems.

Things to focus on:

- Specifying the problem
- Designing the algorithm
- Implementing the algorithm
- Interpreting the result
- Important Technicalities
  - Inputting the problem
  - Ensuring termination

This frame has been deliberately left blank.

Matrix

Definition
*A **matrix** is a rectangular **grid** of things.*
Assume (real) numbers.

Key Point

Linear systems can be **represented** by matrices
with **no** loss of information

The Obligatory Matrix Stunt

$$\begin{aligned} 2x + 3y + z &= -2 \\ 2x - 3y + 4z &= 1 \\ -x + 2y - z &= -2 \end{aligned}$$

The Rules

- Keep only the numbers
- Include "hidden" 0s, 1s, and -1s:

$$\text{“ ”} = 0x, \qquad x = 1x, \qquad -x = -1x$$

- Keep watch for signs

Two Matrices

- **Coefficient** matrix: left-hand side of "$=$"s
- **Augmented** matrix: everything

Why? Often vary right-hand side but keep left-hand side same.
Notation:

$$\begin{bmatrix} 2 & 3 & 0 \\ 3 & -3 & 4 \\ -1 & 2 & -1 \end{bmatrix} \qquad \left[\begin{array}{ccc|c} 2 & 3 & 0 & -2 \\ 2 & -3 & 4 & 1 \\ -1 & 2 & -1 & -2 \end{array}\right]$$

Summary

- Use **Gaussian elimination** to solve linear systems
- Result is in **echelon form**
- **Reduced** echelon form is guaranteed unique
- **Matrices** encode linear systems
- Can perform GE on matrix rather than linear system