TMA4115 Matematikk 3

Andrew Stacey

Norges Teknisk-Naturvitenskapelige Universitet
Trondheim

Spring 2010

Lecture 9: What's The Difference?

Andrew Stacey

Norges Teknisk-Naturvitenskapelige Universitet
Trondheim

10th February 2010

Key Points

- Focus on **solving problem**
- Find **useful** description of solution set
- Use **reversible** techniques

Linear Systems

Linear $\Longrightarrow$ constants + multiples of x, y, z (etc.)

Linear $\Longrightarrow$ no funny functions!

$$\begin{aligned} 2x + \quad & 3y + \quad 4z = 7 && \checkmark \\ 2x^2 + \quad & 3\sqrt{y} + 4\sin(z) = 7 && \times \\ \sin(2)x + \log(3)y + \quad & e^4 z = 7 && \checkmark \end{aligned}$$

Linear system = family of linear equations.

Key Question
What do we do with them?

Answer
Solve them!

Solving Linear Systems

Solve

$$-x + 4y = 9$$
$$3x + 2y = 1$$

Question

What does that mean?

Answer

- Find two numbers which when put in for x and y make the two statements true.
- Find all possible such pairs, or show that none exist.

ODEs and Linear Systems

Similarities

- Focus on **solving**

$\Longrightarrow$ Method subordinate to goal

- In practise, use approximation methods

Differences

- Theoretically possible to solve exactly

Since You Mentioned Them...

Linear Systems From ODEs

The final stage in solving an IVP is a **linear system**

Linear Systems from ODEs

Solve

$$y'' + 2y' + y = 0, \qquad y(0) = 1, y'(0) = 0$$

- Solutions: e^{-x}, xe^{-x}
- General Solution: $ae^{-x} + bxe^{-x}$
- Fit to IVP:

$$ae^{-0} + b0e^{-0} = 1$$
$$-ae^{-0} - b0e^{-0} + be^{-0} = 0$$

- Rearrange:

$$a + 0b = 1$$
$$-a + b = 0$$

- Read off solution: $a = 1$, $b = 1$
- Interpret correctly: solution of IVP is $e^{-x} + xe^{-x}$

Solving Linear Systems: Key Step

Solution of
$$\begin{aligned} -x + 4y &= 9 \\ 3x + 2y &= 1 \end{aligned}$$

is solution of and **vice versa**
$$\begin{aligned} -x + 4y &= 9 \\ 14y &= 28 \end{aligned}$$

is solution of and **vice versa**
$$\begin{aligned} -x + 4y &= 9 \\ y &= 2 \end{aligned}$$

is solution of and **vice versa**
$$\begin{aligned} -x + 8 &= 9 \\ y &= 2 \end{aligned}$$

is solution of and **vice versa**
$$\begin{aligned} x &= -1 \\ y &= 2 \end{aligned}$$

Which Is Easier?

1. Solve
$$\begin{aligned} -x + 4y &= 9 \\ 3x + 2y &= 1 \end{aligned}$$

2. Solve
$$\begin{aligned} -x + 8 &= 9 \\ y &= 2 \end{aligned}$$

Key Idea

Solutions are **same** so transform **first** into **second**

Secondary Goal

Transform:
linear systems of 2nd type into linear systems of 1st type.

Simple Steps: Elementary Operations

1. Scaling: $\lambda \neq 0$

$$\begin{aligned} a_1x_1 + a_2x_2 + \ldots + a_nx_n &= b \\ \downarrow & \\ \lambda a_1x_1 + \lambda a_2x_2 + \ldots + \lambda a_nx_n &= \lambda b \end{aligned}$$

2. Swapping:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \downarrow & \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \end{aligned}$$

Simple Steps: Elementary Operations

3. Adding: any $\lambda \in \mathbb{R}$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \downarrow & \\ (a_{11}+\lambda a_{21})x_1 + (a_{12}+\lambda a_{22})x_2 + \ldots + (a_{1n}+\lambda a_{2n})x_n &= b_1+\lambda b_2 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \end{aligned}$$

Finding Solutions

- In each case: if $x_1, x_2, \ldots, x_n$ satisfy 1st set, they **also** satisfy 2nd set.
- Scaling: $c = d \Longrightarrow \lambda c = \lambda d$
- Swapping: $\big(a = b \text{ and } c = d\big) \Longrightarrow \big(c = d \text{ and } a = b\big)$
- Adding: $\big(a = b \text{ and } c = d\big) \Longrightarrow \big(a + \lambda c = b + \lambda d \text{ and } c = d\big)$

$\Longrightarrow$ in each case, solution of 1st set **is** solution of 2nd.

"$\Longleftrightarrow$" comes from **reversibility**

Each step can be **reversed** by a similar step.

Examples

$$\begin{array}{rcl} 2x+3y+z=-2 & \xrightarrow{R_2-R_1} & 2x+3y+z=-2 \\ 2x-3y+4z=1 & & -6y+3z=3 \\ -x+2y-z=-2 & & -x+2y-z=-2 \end{array}$$

$$\xrightarrow{R_3+\frac{1}{2}R_1} \begin{array}{rcl} 2x+3y+z=-2 \\ -6y+3z=3 \\ 3\frac{1}{2}y-\frac{1}{2}z=-3 \end{array}$$

$$\xrightarrow{R_3+\frac{7}{12}R_2} \begin{array}{rcl} 2x+3y+z=-2 \\ -6y+3z=3 \\ 1\frac{1}{4}z=-1\frac{1}{4} \end{array}$$

Stop here. Read off solution: $z = -1$, so $y = -1$, so $x = 1$

Slightly Simpler

$$\begin{array}{rcl} 2x+3y+z=-2 & \xrightarrow{R_1\leftrightarrow R_3} & -x+2y-z=-2 \\ 2x-3y+4z=1 & & 2x-3y+4z=1 \\ -x+2y-z=-2 & \xrightarrow{R_3+2R_1} & 2x+3y+z=-2 \end{array}$$

$$\xrightarrow{R_2+2R_1} \begin{array}{rcl} -x+2y-z=-2 \\ y+2z=-3 \\ 7y-z=-6 \end{array}$$

$$\xrightarrow{R_3+7R_2} \begin{array}{rcl} -x+2y-z=-2 \\ y+2z=-3 \\ -15z=15 \end{array}$$

Difference? No nasty fractions!

Summary

- Solving linear systems means finding numbers that make them **true**
- Some linear systems are easier than others
- Elementary operations change linear systems **without** changing solutions
- Choose to make **simpler**